Weakly Ore type condition $d(x) + d(y) \geq n - 1$ for vertex pancyclicity

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Abstract: Let G be a graph of order n. For graph to be Hamiltonian beginning with Dirac's classic result (Proc.London Math.Soc.2 (1952), 69-81), Dirac's Theorem was followed by that of Ore (Amer.Math.Monthly 67(1960),55). In 1971 Bondy considered Ore condition: $d(x) + d(y) \geq n$ for pancyclic and proved that if $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices x, y, then G is pancyclic or $G \in K_{n/2,n/2}$ (J.Combin.Theory Ser.B 11(1971), 80-84). In 1985 Ainouche and Christofides considered $d(x) + d(y) \geq n - 1$ for Hamiltonian and obtained that if $d(x) + d(y) \geq n - 1$ for every pair of nonadjacent vertices x, y, then G is Hamiltonian or $K^C_{(n+1)/2} \vee G_{(n-1)/2}$ (J. London Math.Soc. 32, 385-391). In 1994 Aldred, Holton and Zhang studied pancyclic and proved that if $d(x) + d(y) \geq n - 1$ for every pair of nonadjacent vertices x, y, then G is pancyclic or $G \in \{K^C_{(n+1)/2} \vee G_{(n-1)/2}, K_{n/2,n/2}\}$ (Discrete Math.127,23-29). In this note we investigate vertex-pancyclic and obtain that if $d(x) + d(y) \geq n - 1$ for every pair of nonadjacent vertices x, y, then G is vertex 4-pancyclic or $G \in \{K^C_{(n+1)/2} \vee G_{(n-1)/2}, K_{n/2,n/2}, K^c_2 \vee (K_1 \cup K_{n-3}), K_1 \vee K^1_3 : K_{n-4}, \}$.

Key words: Pancyclic graphs; Vertex pancyclic graphs; Ore type condition

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1 Introduction

We consider finite, undirected, and simple graph G with the vertex set $V(G)$ and the edge set $E(G)$. The complete graph of order n is denoted by K_n and the empty graph of order n is denoted by K^C_n. The complete bipartite graph with the partite sets A and B with $|A| = p$ and $|B| = q$ is denoted by $K_{p,q}$. We denote by $\delta(G)$ (or δ) the minimum degree. If H and S are subsets of $V(G)$ or subgraphs of G, we denote by $N_H(S)$ the set of vertices in H which are adjacent to some vertex in S and set $|N_H(S)| = d_H(S)$. In particular, when $H = G$ and $S = \{u\}$, then let $N_G(S) = N(u)$ and set $d_G(S) = d(u)$ and $N[u] = N(u) \cup \{u\}$. We denote by $G - H$ and $G[S]$ the induced subgraphs of G on $V(G) - V(H)$ and S, respectively. Let $C_m = x_1x_2 \ldots x_mx_1$ denote a cycle of order m. Define

$N^+_{C_m}(u) = \{x_{i+1} : x_i \in N_{C_m}(u)\}$ and $N^-_{C_m}(u) = \{x_{i-1} : x_i \in N_{C_m}(u)\}$, $N^\pm_{C_m}(u) = N^+_{C_m}(u) \cup N^-_{C_m}(u)$, where subscripts are taken modulo m.

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A graph G of order n is said to be Hamiltonian if G contains cycle of length n. And a graph G is said to be r-pancyclic if G contains a cycle of length k for each k such that $r \leq k \leq n$. 3-pancyclic short for pancyclic. A vertex of a graph G is r-pancyclic if it is contained in a cycle of length k for every k between r and n, and graph G is vertex r-pancyclic if every vertex is r-pancyclic, vertex 3-pancyclic short for vertex pancyclic.

We mention some fundamental results in order to increase generality.

Theorem 1.1 *(Dirac,1952 [1]) If G is a graph of order n and $\delta \geq n/2$,then G is Hamiltonian.*

Theorem 1.2 *(Ore,1960 [2]) If G is a connected graph of order $n \geq 3$, $d(x) + d(y) \geq n$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G is Hamiltonian..*

Theorem 1.3 *(Erdős,1962 [6]) Let G be a graph of order n and size m. If $\delta \geq n/2$ or $m \geq \max\{C^2_{n-\delta} + \delta^2, C^2_{(n+2)/2} + [(n-1)/2]^2\}$, then G is Hamiltonian.*

Theorem 1.4 *(Erdős and Gallai,1959 [7]) Let G be a graph of order n and size m. If $m \geq n(n-1)/2$, then G is Hamiltonian.*

In 1971 Bondy [3] obtained the following results on pancyclicity with Ore condition and graph size.

Theorem 1.5 *(Bondy,1971 [3]) If G is a 2-connected graph of order $n \geq 3$, $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices $x, y \in V(G)$, then G is pancyclic or $G = K_{n/2,n/2}$.*

Theorem 1.6 *(Bondy,1971 [3]) If G is a Hamiltonian of order n and size $m \geq n^2/4$, then G is pancyclic or $G = K_{n/2,n/2}$.*

In 1985 Ainouche and Christofides [4] considered $d(x) + d(y) \geq n - 1$ for Hamiltonian and obtained:

Theorem 1.7 *(Ainouche and Christofides,1985 [4]) If G is a 2-connected graph of order $n \geq 3$, $d(x) + d(y) \geq n - 1$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G is Hamiltonian or $K^C_{(n+1)/2} \vee G_{(n-1)/2}$.*

Theorem 1.8 *(Bollobás and Brightwell,1993 [4]) If G is a graph on n vertices and $W \subseteq V(G)$, and $d(x) + d(y) \geq n$ for each pair of nonadjacent vertices $x, y \in W$, then G has a cycle containing all the vertices of W.*

In 1994 Aldred, Holton and Zhang [5] relaxed Ore' condition for pancyclic graphs by considered condition $d(x) + d(y) \geq n - 1$ and obtained:

Theorem 1.9 *(Aldred, Holton and Zhang [5] or Theorem 36 of survey [6]) If G is a 2-connected graph of order $n \geq 3$, $d(x)+d(y) \geq n-1$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G is pancyclic or $G \in \{K^C_{(n+1)/2} \vee G_{(n-1)/2}, K_{n/2,n/2}, C_5\}$.*

Theorem 1.10 *((Hendry [5] or see Corollary 7 in [11]) Let G be a graph of order $n \geq 3$ with $\delta \geq (n+1)/2$, then G is vertex pancyclic.*

The following vertex pancyclic result is the Corollary 12 in Ref. [11] and Theorem 1.5 in Ref. [12].

Theorem 1.11 *(Randerath et al.[11] or Zhang et al.[12]) If G is a 2-connected graph of order $n \geq 3, d(x)+d(y) \geq n$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G is vertex 4-pancyclic or $G = K_{n/2,n/2}$.*

Now, we consider weakly Ore type condition $d(x)+d(y) \geq n-1$ for vertex pancyclic and obtain the following result.

Theorem 1.12 *If G is a 2-connected graph of order $n \geq 7$, $d(x)+d(y) \geq n-1$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G is vertex 4-pancyclic or $G \in \{G_{(n-1)/2} \vee K^C_{(n+1)/2}, K_{n/2,n/2}, K^c_2 \vee (K_1 \cup K_{n-3}), K_1 \vee K^1_3 : K_{n-4}\}$.*

Where $G_{(n-1)/2}$ is a subgraph of order $(n-1)/2, G_{(n-1)/2} \vee K^C_{(n+1)/2}$ is used to denote the graph obtained by taking the join of $G_{(n-1)/2}$ and $K^C_{(n+1)/2}$. $K^c_2 \vee (K_1 \cup K_{n-3})$ and $K_1 \vee K^1_3 : K_{n-4}$ can be found in Lemma 2.3.

Note that: Under the condition $d(x)+d(y) \geq n-1$, when connectivity $\kappa = 1$, clearly then graph G is the graph consisting of two complete graphs joined at a point.

In Section 2 we discussion graphs of order $n = 4, 5, 6$ and satisfying the condition $d(x)+d(y) \geq n-1$ in Lemma 2.5.

2 The proof of Theorem

The proof will be divided into lemmas. It is readily seen that the Theorem 1.12 follows from Lemma 2.2, 2.3, 2.4, and Theorem 1.7.

Lemma 2.1 *Let $C_m = x_1x_2 \ldots x_mx_1$ be a cycle length m of graph G, if there does not exist C_{m-2} containing v in G, then for any i with $1 \leq i \leq m$, we obtain the following are all true: (1). When $v \notin \{x_{i+1}, x_{i+2}\}$, then $x_ix_{i+3} \notin E(G)$. (2): When $v \notin \{x_{i+1}, x_{i+2}, x_{i+3}\}$, then $N_{G-C_m}(x_i) \cap N_{G-C_m}(x_{i+4}) = \emptyset$. (3): If $v \notin \{x_{i+1}\}$ and $x_ix_{i+2} \in E(G)$, then when $v \notin \{x_{j+1}\}$ we have $x_jx_{j+2} \in E(G)$ and when $v \notin \{x_{j+1}, x_{j+2}\}$ we have $N_{G-C_m}(x_j) \cap N_{G-C_m}(x_{j+3}) = \emptyset$ for any $j \neq i, i+1$; On other hand, if $v \notin \{x_{i+1}, x_{i+2}\}$ and $N_{G-C_m}(x_i) \cap N_{G-C_m}(x_{i+3}) \neq \emptyset$, then when $v \notin \{x_{j+1}\}$*

we have $x_jx_{j+2} \notin E(G)$ and when $v \notin \{x_{j+1}, x_{j+2}\}$ we have $N_{G-C_m}(x_j) \cap N_{G-C_m}(x_{j+3}) = \emptyset$ for any $j \neq i, i+1, i+2$. (4): If $x_ix_h \in E(G)$, $h \neq i+1, i+2$, then when $v \notin \{x_{i+1}, x_{i+2}\}$ we have $x_{i+3}x_{h+1} \notin E(G)$ and when $v \notin \{x_{i+1}, x_{h+1}\}$ we have $x_{i+2}x_{h+2} \notin E(G)$.

Proof. (1). If $x_ix_{i+3} \in E(G)$, then there exists $C_{m-2} = x_1x_2 \dots x_ix_{i+3} \dots x_mx_1$ in G containing v, a contradiction.

(2). If $u \in N_{G-C_m}(x_i) \cap N_{G-C_m}(x_{i+4})$, then we get $C_{m-2} = x_1x_2 \dots x_iux_{i+4} \dots x_mx_1$ in G containing v, a contradiction.

(3). If $x_ix_{i+2} \in E(G)$ and if there exist $j \neq i, i+1$ with $x_jx_{j+2} \in E(G)$. Without loss of generality, assume $j \geq i$, then we get $C_{m-2} = x_1x_2 \dots x_ix_{i+2} \dots x_jx_{j+2} \dots x_mx_1$ in G containing v, a contradiction. We can apply the similar arguments and obtain the rest of (3) are true.

(4). If $x_ix_h \in E(G)$, where $h \neq i+1, i+2$ and $x_{i+3}x_{h+1} \in E(G)$. Without loss of generality, assume $h \geq i$, then we get $C_{m-2} = x_1x_2 \dots x_ix_hx_{h-1} \dots x_{i+3}x_{h+1}x_{h+2} \dots x_mx_1$ in G containing v, a contradiction. By the similar proof as above, if $x_ix_h \in E(G)$,where $h \neq i+1, i+2$ and $x_{i+2}x_{h+2} \in E(G)$. Without loss of generality, assume $h \geq i$, then we get $C_{m-2} = x_1x_2 \dots x_ix_hx_{h-1} \dots x_{i+2}x_{h+2}$ $x_{h+3} \dots x_mx_1$ in G containing v, a contradiction.

Lemma 2.2 *Let G be a 2-connected graph of order $n \geq 7$, $d(x) + d(y) \geq n-1$ for each pair of nonadjacent vertices $x, y \in V(G)$, if there exists C_m containing v ($m \geq 7$), then G contains C_{m-2} containing v.*

Proof. Assume, to the contrary, that there does not exist C_{m-2} containing v. Then let $C_m = x_1x_2 \dots x_mx_1$, we consider the following cases.

Case 1. $m = 7$.

Let $C_7 = x_1x_2x_3x_4x_5x_6x_7x_1$ without loss of generality, assume $v = x_1$.

In this case, clearly $x_1x_4, x_1x_5 \notin E(G)$ (Otherwise, if $x_1x_4 \in E(G)$, we can obtain C_5 : $x_1x_4x_5x_6x_7x_1$, a contradiction. If $x_1x_5 \in E(G)$, we can obtain C_5 : $x_1x_2x_3x_4x_5x_1$, a contradiction).

Now, we consider the following two subcases.

Subcase 1.1. $x_1x_3, x_2x_4 \notin E(G)$.

In this case, if $x_4x_6 \notin E(G)$, then clearly we can check that $d_{C_7}(x_1) + d_{C_7}(x_4) \leq 5$. If $x_4x_6 \in E(G)$, then $x_1x_6 \notin E(G)$ (Otherwise, if $x_4x_6 \in E(G)$, then we can obtain C_5 : $x_1x_2x_3x_4x_6x_1$, a contradiction.). Thus, we also have $d_{C_7}(x_1) + d_{C_7}(x_4) \leq 5$.

And we have that both x_1 and x_4 do not have any common neighbor vertex in $G - C_7$ (Otherwise, if $x \in V(G - C_7)$ with $x_1x, x_4x \in E(G)$, then we obtain a $C_5 : xx_1x_2x_3x_4x$, a contradiction.). Hence we can check that $d(x_1) + d(4) \leq n-2$, this contradicts the condition of lemma 2.2.

Subcase 1.2. x_1x_3 or $x_2x_4 \in E(G)$.

In this case, we have $x_1x_6, x_5x_7 \notin E(G)$ (Otherwise, we obtain a C_5: containing v, a contradiction.). By considering $C_7 = x_1x_7x_6x_5x_4x_3x_2x_1$ so we can apply the similar arguments as subcase

1.1, and we have $d(x_1)+d(5) \le n-2$, this contradicts the condition of lemma 2.2.

Case 2. $m \ge 8$.

Let $C_m = x_1x_2x_3x_4\dots x_mx_1$ without loss of generality, assume $v = x_1$. Then we consider the following two subcases.

Subcase 2.1. x_1x_3 or $x_1x_{m-1} \in E(G)$.

Without loss of generality, assume $x_1x_3 \in E(G)$. In this case, we have $x_3x_6 \notin E(G)$ and both x_3 and x_6 do not have any common neighbor vertex in $G - C_m$ (Otherwise, if $x_3x_6 \in E(G)$, then we get $C_{m-2} : x_3x_6x_7\dots x_3$ containing x_1, a contradiction. If $u \in V(G - C_m)$ is adjacent to both x_3 and x_6, we also can obtain a $C_{m-2} = x_3ux_6x_7\dots x_mx_1x_3$ containing x_1, a contradiction.). Then we consider the following cases.

(1). If $x_3x_7 \notin E(G)$.

When $x_r \in \{x_1, x_2, \dots, x_m\} \setminus \{x_5\}$ is adjacent to x_6, then x_{r-1} is not adjacent to x_3 (Otherwise, we can obtain a $C_{m-2} = x_3x_{r-1}x_{r-2}\dots x_6x_rx_{r+1}\dots x_3$ containing x_1, a contradiction). Since x_4, x_6, x_8 are not adjacent to x_6, and x_3, x_5, x_7 are not adjacent to x_3, respectively.

Hence we have $d_{C_m}(x_3) \le m - |N_{C_m}(x_6) \setminus \{x_5\}| - |\{x_3, x_5, x_7\}| \le m - d_{C_m}(x_6) - 2$, this implies

$$d_{C_m}(x_3) + d_{C_m}(x_6) \le m - 2 \tag{1}$$

Since both x_3 and x_6 do not have any common neighbor vertex in $G - C_m$. Hence we have

$$d_{G-C_m}(x_3) + d_{G-C_m}(x_6) \le |V(G - C_m)| \tag{2}$$

By the inequalities (1) and (2), we have

$d(x_3) + d(x_6) \le |V(G - C_m)| + m - 2 \le n - 2$, this contradicts the condition of lemma 2.2.

(2) If $x_3x_7 \in E(G)$. Then we have $x_5x_9 \notin E(G)$ (Otherwise, if $x_5x_9 \in E(G)$, we can obtain $C_{m-2} : x_3x_7x_6x_5x_9x_{10}\dots x_3$ containing x_1, a contradiction. Where if $m = 8$, then $x_9 = x_1$). Then by a similar arguments as above inequality (1), we can check

$$d_{C_m}(x_5) + d_{C_m}(x_8) \le m - 2 \tag{3}$$

Clearly, both x_5 and x_8 do not have any common neighbor vertex in $G - C_m$ (Otherwise, if $u \in V(G - C_m)$ is adjacent to both x_5 and x_8, then we get $C_{m-2} = x_5ux_8x_9\dots x_1x_3x_4x_5$ containing x_1, a contradiction. Where if $m = 8$, then $x_9 = x_1$). Hence we have

$$d_{G-C_m}(x_5) + d_{G-C_m}(x_8) \le |V(G - C_m)| \tag{4}$$

Clearly $x_5x_8 \notin E(G)$(Otherwise, if $x_5x_8 \in E(G)$, then we get $C_{m-2} = x_5x_8x_9\dots x_5$ containing x_1, a contradiction. Where if $m = 8$, then $x_9 = x_1$).

Combining inequalities (3) and (4), we have

$d(x_5) + d(x_8) \leq |V(G - C_m)| + m - 2 \leq n - 2$, this contradicts the condition of lemma 2.2.

Subcase 2.2. $x_1x_3, x_1x_{m-1} \notin E(G)$.

In this case, we consider the following .

(1). $x_ix_{i+2} \in E(G)$ for some $i \in \{2, 3, 4\}$.

Clearly, both x_1 and x_{m-2} do not have any common neighbor vertex in $G - C_m$(Otherwise, if $u \in V(G-C_m)$ is adjacent to both x_1 and x_{m-2}, then we get $C_{m-2} = x_1x_2 \ldots x_ix_{i+2}x_{i+3} \ldots x_{m-2}ux_1$ containing x_1, a contradiction.). Hence we have

$$d_{G-C_m}(x_5) + d_{G-C_m}(x_8) \leq |V(G - C_m)| \tag{5}$$

Clearly $x_1x_{m-2} \notin E(G)$(Otherwise, if $x_1x_{m-2} \in E(G)$, then we get $C_{m-2} = x_1x_2 \ldots x_{m-2}x_1$ containing x_1, a contradiction). Then by a similar arguments as above inequality (1), we can check

$$d_{C_m}(x_1) + d_{C_m}(x_{m-2}) \leq m - 2 \tag{6}$$

Combining inequalities (5) and (6), we have

$d(x_5) + d(x_8) \leq |V(G - C_m)| + m - 2 \leq n - 2$, this contradicts the condition of lemma 2.2.

(2). $x_ix_{i+2} \in E(G)$ for some $i \in \{5, 6, \ldots, m - 2\}$.

In this case, we replace x_1, x_{m-2} by x_1, x_4 and we can apply the similar arguments as above (1) of Subcase 2.2, and obtain a contradiction.

(3). $x_ix_{i+2} \notin E(G)$ for any $i \in \{1, 2, \ldots, m - 1\}$.

In this case, we consider the following subcases.

$(3-1)$. Both x_1 and x_4 do not have any common neighbor vertex in $G - C_m$.

In this case, we have $x_1x_4 \notin E(G)$ (Otherwise, if $x_1x_4 \in E(G)$, then we get $C_{m-2} : x_1x_4x_5 \ldots x_m$ x_1 containing x_1, a contradiction).

Then, we can check

$d(x_1) + d(x_4) \leq |V(G - C_m)| + m - 2 \leq n - 2$, this contradicts the condition of lemma 2.2.

$(3-2)$. Both x_1 and x_4 have one common neighbor vertex in $G - C_m$. Then both x_5 and x_8 do not have any common neighbor vertex in $G - C_m$. (Otherwise, let $u \in V(G - C_m)$ is adjacent to both x_1 and x_4. If $v \in V(G - C_m)$ is adjacent to both x_5 and x_8. (i). When $u = v$, then we obtain a $C_{m-2} = x_1ux_5x_6 \ldots x_1$ containing x_1, a contradiction. (ii). When $u \neq v$, then we obtain a $C_{m-2} = x_1ux_4x_5vx_8x_9 \ldots x_1$ containing x_1, a contradiction.).

Hence, by the similar arguments as above we have

$d(x_4) + d(x_8) \leq |V(G - C_m)| + m - 2 \leq n - 2$, this contradicts the condition of lemma 2.2.

Lemma 2.3 *Let G be a 2-connected graph of order $n \geq 7$, $d(x) + d(y) \geq n - 1$ for each pair of nonadjacent vertices $x, y \in V(G)$, if there does not exist C_4 containing v, then $G \in \{K_2^c \vee (K_1 \cup K_{n-3}), K_1 \vee K_3^1 : K_{n-4}\}$*

proof. For some vertex v, if there does not exist C_4 containing v. Then induced subgraph $G[N(v)]$ does contain any a path of order 3, and not two vertices of $N(v)$ that have a common neighbor vertex in $G - N[v]$, then we claim that $|N(v)| \leq 3$. Otherwise, if $|N(v)| \geq 4$, then let $x, y \in N(v)$ with $xy \notin E(G)$, then we can check that $d(v) + d(u) \leq n - 2$, a contradiction.

When $|N(v)| = 2$. In this case, clearly $G - N[v]$ is a complete subgraph. Otherwise, if $x, y \in V(G - N[v])$ with $xy \notin E(G)$, then we can check that $d(v) + d(x) \leq n - 2$, a contradiction. Hence we have $G = K_2^c \vee (K_1 \cup K_{n-3})$

When $|N(v)| = 3$. In this case, clearly $N(v)$ have adjacent two vertices (Otherwise, let $x, y \in N(v)$, then we can check that $d(x) + d(y) \leq n - 2$, a contradiction). Without loss of generality, say $x, y \in N(v)$ with $xy \in E(G)$, Then we have (1). Each of $\{x, y\}$ is adjacent to only one vertex of $G - N[v]$ (Otherwise, if x is adjacent to at least two vertices of $G - N[v]$, let $w = N(v) \setminus \{x, y\}$, then we can check that $d(y) + d(w) \leq n - 2$, a contradiction). We also have $G - N[v]$ is a complete subgraph. In this case, we have $G = K_1 \vee K_3^1 : K_{n-4}$.

Lemma 2.4 *Let G be a 2-connected Hamiltonian graph of order $n \geq 7$, $d(x) + d(y) \geq n - 1$ for each pair of nonadjacent vertices $x, y \in V(G)$, for any vertex v, if there is not C_{n-1} containing v in G, then $G \in \{K_{n/2,n/2}, K_{n/2,n/2} - e\}$.*

Proof. Assume, to the contrary, that $G \notin \{K_{n/2,n/2}, K_{n/2,n/2} - e\}$, then we have the following claims.

Claim 1. $x_i x_{i+3} \in E(G)$ or $x_{i-1} x_{i+2} \in E(G)$ for every $x_i \in V(C_n)$ with $v \notin \{x_{i-1}, x_{i+1}, x_{i+3}\}$

Since not C_{n-1}, then when $x_h \in C_n$ is adjacent to x_{i+2}, then x_{h-1} is not adjacent to x_i. Namely none of $N^-_{C_n}(x_{i+2})$ are adjacent x_i. Assume that Claim 1 is not true, then $x_i x_{i+3}, x_{i-1} x_{i+2} \notin E(G)$. Together with $x_i x_{i-2}, x_{i+2} x_{i+4} \notin E(G)$ (Otherwise, if $x_i x_{i-2} \in E(G)$, then $x_{i-2} x_i x_{i+1} x \ldots x_{i-2} = C_{n-1}$, a contradiction. If $x_{i+2} x_{i+4} \in E(G)$, then $x_{i+2} x_{i+4} x_{i+5} \ldots x_{i+2} = C_{n-1}$, a contradiction). Hence we can check

$|N_{C_n}(x_i)| \leq |V(G)| - |N^-_{C_n}(x_{i+2})| - |\{x_{i+3}, x_{i-2}\}|$, this implies $d(x_i) + d(x_{i+2}) \leq n - 2$, this contradicts the condition of lemma 2.4. The contradicts shows that claim 1 is true.

Claim 2. $x_i x_j \notin E(G)$ or $x_i x_{j+1} \notin E(G)$ for every vertex x_i and x_j in C_n with $v \neq x_i$.

Otherwise, if there exists x_j in C_n such that $x_i x_j, x_i x_{j+1} \in E(G)$. By Claim 1 we have $x_{i-1} x_{i+2} \in E(G)$ or $x_{i-2} x_{i+1} \in E(G)$. Then we have $x_{i-1} x_{i+2} x_{i+3} \ldots x_j x_i x_{j+1} x_{j+2} \ldots x_{i-1} = C_{n-1}$ or $x_{i-2} x_{i+1} x_{i+2} \ldots x_j x_i x_{j+1} x_{j+2} \ldots x_{i-2} = C_{n-1}$, respectively, contradiction .

Claim 3. $\{x_1, x_3, x_5, \ldots, x_{2m-1}, \ldots\}$ is a independent set and $\{x_2, x_4, x_6, \ldots, x_{2m}, \ldots\}$ is also a independent set.

Proof of Claim 3. By Claim 2, we know that $d(x_i) \leq n/2$ $(i = 1, 2, \ldots)$ (Otherwise, if $\mathrm{d}(x_i) > n/2$ where $i = 1, 2, \ldots$ and $x_i \neq v$, then there must exist $x_j, x_{j+1} \in V(C_n)$ which all adjacent to x_i, this contradicts Claim 2. If if $d(v) > n/2$, $x_i = v$. Since not C_{n-1}, then $x_i x_{i+2} \notin E(G)$, so we have $d(x_i) + d(x_{i+2}) \geq n - 1$, this implies $d(x_{i+2}) > n/2$. Then there must exist $x_j, x_{j+1} \in V(C_n)$ which all adjacent to x_{i+2}, this contradicts Claim 2).

Since not C_{n-1}, then $x_{i-1}x_{i+1} \notin E(G)$, so we have $d(x_{i-1}) + d(x_{i+1}) \geq n - 1$ $(i = 1, 2, \ldots)$.

This implies $(n - 1)/2 \leq d(x_i) \leq n/2$ $(i = 1, 2, \ldots)$.

Then for every x_i with $x_i \neq v$, since not C_{n-1}, by Claim 2, there do not exist x_j, x_{j+1} in C_n satisfying $x_i x_j, x_i x_{j+1} \in E(G)$. Then we have

Claim (3.1): If there exist x_h, x_{h+1} that are not adjacent to x_i with $x_i \neq v$, then x_i will be adjacent to every vertex of $(\ldots, x_{h-2m+1}, \ldots, x_{h-3}, x_{h-1}, x_{h+2}, x_{h+4}, \ldots, x_{h+2m} \ldots)$.

Claim (3.2). If there do not exist x_h, x_{h+1} that are not adjacent to x_i with $x_i \neq v$, then when i is even, x_i will be adjacent to every vertex of $\{x_1, x_3, x_5, \ldots, x_{2m-1}, \ldots\}$; When i is odd, x_i will be adjacent to every vertex of $\{x_2, x_4, x_6, \ldots, x_{2m}, \ldots\}$.

(1). If there do not exist Claim (3.1).

$(1-1)$. When $x_i = v$ is not adjacent to any vertex of $\{\ldots, x_{i-4}, x_{i-2}, x_{i+2}, x_{i+4}, \ldots\}$. Then Claim 3 holds.

$(1-2)$. When $x_i = v$ is adjacent to some vertex of $\{\ldots, x_{i-4}, x_{i-2}, x_{i+2}, x_{i+4}, \ldots\}$.

For example, if $x - i = v$ is adjacent to x_{i+2r}. Then x_{i+1} will be adjacent to x_{i+2r+2} or x_{i+1} will be adjacent to x_{i+2r-2}. Then we can get a C_{n-1} containing x_i, a contradiction.

(2). If there exist Claim (3.1).

$(2-1)$. If there also exist Claim (3.2). Then clearly there exist x_j, x_{j+1} such that one vertex satisfying Claim (3.1) and another vertex satisfying Claim (3.2), this implies that both x_j and x_{j+1} will have at least two common vertices of C_n. Then by Claim 2, we have a C_{n-1} containing x_i, a contradiction.

$(2-2)$. If there do not exist Claim (3.2).

In this case, By Claim 2, we have that if both x_h and x_{h+1} are not adjacent to x_i; then both x_{h+2} and x_{h+1} or both x_h and x_{h-1} are not adjacent to x_{i+1}(Otherwise, both x_i and x_{i+1} will have at least two common vertices of C_n. Then get a C_{n-1} containing x_i.).

In this case, we choose $x_i = v$, then by Claim 1 we have $x_{i-1}x_{i+2} \in E(G)$ or $x_i x_{i+3} \in E(G)$. Then we can obtain a C_{n-1} containing x_i, a contradiction.(For example, if $x_{i-1}x_{i+2} \in E(G)$. Since both x_h and x_{h+1} are not adjacent to x_i, and both x_{h+2} and x_{h+1} or both x_h and x_{h-1} are not adjacent to x_{i+1}. Without loss of generality, assume both x_{h+2} and x_{h+1} are not adjacent to x_{i+1}, then we get $C_{n-1} = x_h x_{h-1} \ldots x_{i+2} x_{i-1} x_{i-2} \ldots x_{h+2} x_i x_{i+1} x_{h+3} x_h$).

By the above (1) and Claim (3.2), we have that $\{x_1, x_3, x_5, \ldots, x_{2m-1}, \ldots\}$ is a independent set and $\{x_2, x_4, x_6, \ldots, x_{2m}, \ldots\}$ is also a independent set. By $(n-1)/2 \leq d(x_i) \leq n/2$ $(i = 1, 2, \ldots)$, we get $G \in \{K_{n/2,n/2}, K_{n/2,n/2} - e\}$.

Therefore, Lemma 2.4 is proved.

Lemma 2.5 *If G is a 2-connected graph of order $n = 4, 5, 6$, $d(x)+d(y) \geq n-1$ for each pair of non-adjacent vertices $x, y \in V(G)$, then G is vertex 4-pancyclic or $G \in \{K^C_{(n+1)/2} \vee G_{(n-1)/2}, K_{n/2,n/2}, C_6 + 2e, C_5 + e, C_5\}$.*

proof. Under the condition of Lemma 2.5, by Theorem 1.7, if $G \notin K^C_{(n+1)/2} \vee G_{(n-1)/2}$, then G is a Hamiltonian. Then we consider v-4-pancyclic.

When $n = 4$. Clearly G is vertex 4-pancyclic.

When $n = 5$. Let $C_5 = x_1x_2x_3x_4x_5x_1$ without loss of generality, assume $v = x_1$. If there does not exist C_4 containing x_1, then we easy to see that $G = C_5$ or $G = C_5 + x_2x_5$.

When $n = 6$. Let $C_6 = x_1x_2x_3x_4x_5x_6x_1$ without loss of generality, assume $v = x_1$.

(1). When $d(x_1) \geq 3$.

$(1-1)$. If $x_1x_3 \in E(G)$. Then there exist C_5 containing x_1. $(1-1-1)$. If $x_4x_6 \notin E(G)$, Then there exist C_4 containing x_1. Thus, G is vertex 4-pancyclic. $(1-1-2)$. If $x_4x_6 \notin E(G)$, by $d(x_4) + d(x_6) \geq n-1$, this implies that $d(x_4) \geq 3$ or $d(x_6) \geq 3$. Without loss of generality, assume $d(x_4) \geq 3$. If $x_4x_1 \in E(G)$, Then there exist C_4 containing x_1. Thus, G is vertex 4-pancyclic. If $x_4x_2 \in E(G)$, Then there exist $C_4 = x_1x_3x_4x_2x_1$ containing x_1. Thus, G is vertex 4-pancyclic.

$(1-2)$. If $x_1x_5 \in E(G)$. By considering $C_6 = x_1x_6x_5x_4x_3x_2x_1$ so we can apply the similar arguments as above $(1-1)$. Thus, G is vertex 4-pancyclic.

$(1-3)$. If $x_1x_4 \in E(G)$. In this case, there exist C_4 containing x_1. $(1-3-1)$. If $x_2x_5 \notin E(G)$ or $x_3x_6 \notin E(G)$. Without loss of generality, say $x_3x_6 \notin E(G)$, then by $d(x_3)+d(x_6) \geq n-1$, we have $d(x_3) \geq 3$ or $d(x_6) \geq 3$. This implies $x_3x_5 \in E(G)$ or $x_6x_2 \in E(G)$ or $x_6x_4 \in E(G)$. For example, if $x_6x_2 \in E(G)$, then there exist $C_5 = x_1x_2x_6x_5x_4x_1$ containing x_1. Thus, G is vertex 4-pancyclic. $(1-3-2)$. If $x_2x_5 \in E(G)$ and $x_3x_6 \in E(G)$ with not C_5 containing x_1. Then $G = K_{n/2,n/2}$.

(2). When $d(x_1) = 2$. In this case, by $d(x_1) + d(x_3) \geq n-1$, we have $d(x_3) \geq 3$. This implies $x_3x_5 \in E(G)$ or $x_3x_6 \in E(G)$. $(2-1)$. If $x_3x_6 \in E(G)$, then there exist C_4 containing x_1. Since $d(x_1) = 2$, then $x_1x_4 \notin E(G)$, by $d(x_1) + d(x_4) \geq n-1$, we have $d(x_4) \geq 3$, this implies $x_4x_2 \in E(G)$ or $x_4x_6 \in E(G)$. Hence we easy to see that there exist C_5 containing x_1. Thus, G is vertex 4-pancyclic. $(2-2)$. If $x_3x_6 \notin E(G)$. Since $d(x_1) = 2$, by the condition of Lemma 2.5, we have $d(x_3) \geq 3$, $d(x_4) \geq 3$, $d(x_5) \geq 3$. By $x_3x_6 \notin E(G)$ this implies $x_3x_5 \in E(G)$, hence there exist C_5 containing x_1. If there does not exist C_4 containing x_1, this implies $x_4x_6 \in E(G)$. In this case, $G = C_6 + x_3x_5 + x_4x_6$.

The proof of Lemma 2.5 is complete.

Recent some results concerning cyclability of graphs were obtained. The following result is due to Favaron et al. [9] and Ota [13].

Theorem 2.6 *(Favaron et al.[9] and Ota [13]) Let G be a graph of order n and S is a subset of $V(G)$ with $|S| \geq 3$. If $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices x and y in S, then S is cyclable in G.*

By the work in this paper, one will ask the following problem.

Problem 2.7. *What is the graph G of order n of non-S-cyclable with $d(x) + d(y) \geq n - 1$ for every pair of nonadjacent vertices x and y in S.*

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