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Pan-connectedness of graphs with large neighborhood unions

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Abstract

Let G be a simple graph with n vertices. For any $v \in V(G)$, let $N(v) = \{u \in V(G): uv \in E(G)\}$, $NC(G) = \min\{|N(u) \cup N(v)|: u, v \in V(G)$ and $uv \notin E(G)\}$, and $NC_2(G) = \min\{|N(u) \cup N(v)|: u, v \in V(G)$ and u and v has distance 2 in $E(G)\}$. Let $l \geq 1$ be an integer. A graph G on $n \geq l$ vertices is $[l, n]$-pan-connected if for any $u, v \in V(G)$, and any integer m with $l \leq m \leq n$, G has a (u, v)-path of length m. In 1998, Wei and Zhu (Graphs Combinatorics 14:263–274, 1998) proved that for a three-connected graph on $n \geq 7$ vertices, if $NC(G) \geq n - \delta(G) + 1$, then G is $[6, n]$-pan-connected. They conjectured that such graphs should be $[5, n]$-pan-connected. In this paper, we prove that for a three-connected graph on $n \geq 7$ vertices, if $NC_2(G) \geq n - \delta(G) + 1$, then G is $[5, n]$-pan-connected. Consequently, the conjecture of Wei and Zhu is proved as $NC_2(G) \geq NC(G)$. Furthermore, we show that the lower bound is best possible and characterize all 2-connected graphs with $NC_2(G) \geq n - \delta(G) + 1$ which are not $[4, n]$-pan-connected.

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Pan-connectedness of graphs with large neighborhood unions

Kewen Zhao

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Abstract Let G be a simple graph with n vertices. For any $v \in V(G)$, let $N(v) = \{u \in V(G) : uv \in E(G)\}$, $NC(G) = \min\{|N(u) \cup N(v)| : u, v \in V(G)$ and $uv \notin E(G)\}$, and $NC_2(G) = \min\{|N(u) \cup N(v)| : u, v \in V(G)$ and u and v has distance 2 in $E(G)\}$. Let $l \geq 1$ be an integer. A graph G on $n \geq l$ vertices is $[l, n]$-pan-connected if for any $u, v \in V(G)$, and any integer m with $l \leq m \leq n$, G has a (u, v)-path of length m. In 1998, Wei and Zhu (Graphs Combinatorics 14:263–274, 1998) proved that for a three-connected graph on $n \geq 7$ vertices, if $NC(G) \geq n - \delta(G) + 1$, then G is $[6, n]$-pan-connected. They conjectured that such graphs should be $[5, n]$-pan-connected. In this paper, we prove that for a three-connected graph on $n \geq 7$ vertices, if $NC_2(G) \geq n - \delta(G) + 1$, then G is $[5, n]$-pan-connected. Consequently, the conjecture of Wei and Zhu is proved as $NC_2(G) \geq NC(G)$. Furthermore, we show that the lower bound is best possible and characterize all 2-connected graphs with $NC_2(G) \geq n - \delta(G) + 1$ which are not $[4, n]$-pan-connected.

Keywords Pan-connected graphs · Neighborhood unions

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1 Introduction

We consider finite, undirected simple graphs in this note. Undefined notations and terminology will follow those in [1]. Let G be a graph. As in [1], $\kappa(G)$ and $\delta(G)$

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denote the *connectivity* and the *minimum degree* of G, respectively. If H is a subgraph of G and $v \in V(G)$, then the *neighborhood of* v *in* H, is defined as $N_H(v) = \{u \in V(H) : uv \in E(G)\}$. We further denote $N_G[v] = N_G(v) \cup \{v\}$. A path $x_0x_1 \cdots x_m$ is also referred to as an (x_0, x_m)-path of length m. For $u, v \in V(G)$, the *distance* between u and v in G, denoted $d_G(u, v)$, is the length of a shortest (u, v)-path. The set $N_G(v)$ is sometimes denoted as $N(v)$ and $d_G(u, v)$ as $d(u, v)$, when G is understood in the context. Let $P = (u, v)$ denote a path in the direction from u to v in G and $x \in V(P)$. We denote by x^+ its successor if $x \neq v$ and x^- its predecessor if $x \neq u$. Let $w \in V(G)$ and $N_P^+(w) = \{w^+ : w \in V(P) - \{v\}\}$ and $N_P^-(w) = \{w^- : w \in V(P) - \{u\}\}$. Suppose that $T = x_jx_{j+1} \cdots x_{j+k}$ is a path. If $x_1, \ldots, x_{j-1}, x_{j+k+1}, \ldots, x_{j+k+t} \in V(G) - V(T)$, and if $x_1 \cdots x_{j-1}x_j$ and $x_{j+k} \cdots x_{j+k+t}$ are paths of G, then $x_1 \cdots x_{j-1}x_jTx_{j+k+1} \cdots x_{j+k+t}$ represent the path $x_1 \cdots x_{j+k+t}$ in G.

For an integer $l \geq 1$, if for any $u, v \in V(G)$ and any integer m with $l \leq m \leq n$, G has a (u, v)-path of length m, then G is $[l, n]$*-pan-connected*. Define $NC(G) = \min\{|N(u) \cup N(v)| : u, v \in V(G)$ and $uv \notin E(G)\}$. The sizes of the neighborhood unions have been used to study hamiltonian graphs and pan-connected graphs. The following theorems have been obtained.

Theorem 1.1 (Faudree et al. [2]) *Let G be a graph with $|V(G)| = n$ and $\kappa(G) \geq 2$. If $NC(G) \geq n - \delta(G)$, then G is hamiltonian.*

Theorem 1.2 (Wei and Zhu [3]) *Let G be a graph with $|V(G)| = n \geq 7$ and $\kappa(G) \geq 3$. If $NC(G) \geq n - \delta(G) + 1$, then G is $[6, n]$-pan-connected.*

In [3], Wei and Zhu conjectured that for a graph G with $|V(G)| = n \geq 7$ and $\kappa(G) \geq 3$, if $NC(G) \geq n - \delta(G) + 1$, then G is $[5, n]$-connected. It is proved in this paper.

Theorem 1.3 *Let G be a graph with $|V(G)| = n \geq 7$ and $\kappa(G) \geq 3$. If $NC(G) \geq n - \delta(G) + 1$, then G is $[5, n]$-pan-connected.*

In fact, we prove a stronger theorem for two-connected graphs in which we characterize the class of all graphs which are not $[4, n]$-pan-connected. Define $NC_2(G) = \min\{|N(u) \cup N(v)| : u, v \in V(G)$ and $d_G(u, v) = 2\}$. Clearly, $NC_2(G) \geq NC(G)$.

Theorem 1.4 *Let G be a 2-connected graph with $|V(G)| = n \geq 7$. If $NC_2(G) \geq n - \delta(G) + 1$, then G is $[4, n]$-pan-connected if and only if $G \notin \{G_1, G_2, G_3\}$ (as in Figs. 1, 2, 3).*

In Fig. 1, K_t $(t \geq 3)$ is a complete graph, $|N_{K_t}(y_0)| \geq 1$, $|N_{K_t}(x_1)| \geq 1$; if $y_0x_1 \notin E(G)$, then for any $w \in V(K_t)$, exactly one of $\{wy_0, wx_1\}$ is in $E(G)$; if $y_0x_1 \in E(G)$, wx_1 and wy_0 are not both in $E(G)$. In Fig. 2, K_t is a complete graph, $N_{K_t}(u_i) = \{t_i\}, i = 1, 2$, $N_{K_t}(x_1) \cap \{t_1, t_2\} = \emptyset$ and x_1 is adjacent to at least two vertices in $V(K_t) - \{t_1, t_2\}$. In Fig. 3, K_t, K_m are complete graphs, $d(x_0) \geq 3$, $d(x_m) \geq 3$ and $N(x_0) \subseteq V(K_t) \cup V(K_m)$, $N(x_0) \subseteq V(K_t) \cup V(K_m)$. In Fig. 4, let $L_1 \cong K_4$ be a graph with $V(L_1) = \{x_0, x_1, x_2, y_0^1\}$, and let $L_2 \cong K_3^C$ be a graph

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Fig. 1 G_1

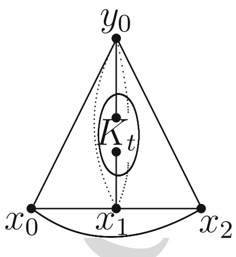


Fig. 2 G_2

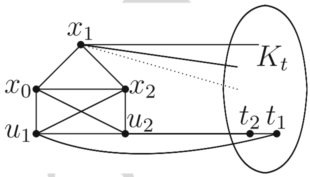


Fig. 3 $G_3, 1 < t \leq m$

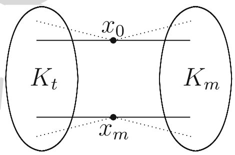
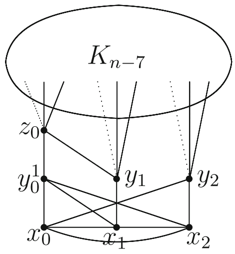


Fig. 4 G_1



with $V(L_2) = \{z_0, y_1, y_2\}$, and let $L_3 \cong K_{n-7}$ with $n - 7 \geq 2$. Assume that all the L_i's are vertex disjoint. Let G_4 be obtained from $L_1 \cup (L_2 \vee L_3)$ by adding four edges $y_0^1 z_0$, $x_1 y_1$, $z_0 y_1$ and $x_2 y_2$. Thus each G_i ($i = 1, 2, 3$) denotes a family of graphs. We also use G_i to denote a particular member in this family.

Clearly, if G is complete, Theorem 1.4 holds. Throughout the following sections of this paper we assume that G is not a complete graph. We shall prove our main theorem by induction. In Sect. 3, we deal with the induction basis and in Sect. 4, we complete the induction step.

2 Lemmas

Let $P_m = x_0 x_1 \cdots x_m$ be an (x, y)-path of length m in G, where $x = x_0$ and $y = x_m$ are called the *ends*, $x_1, x_2, \ldots, x_{m-1}$ are called the *inner vertices*. Throughout the following sections we assume that G is a 2-connected graph with $|V(G)| = n \geq 7$

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such that

$$NC_2(G) \geq |V(G)| - \delta(G) + 1 = n - \delta + 1. \quad (1)$$

If $\delta(G) = 2$, then $NC_2(G) \geq n - 1$. Since G is not complete, $\exists u, v \in V(G)$ such that $d(u, v) = 2$. Clearly $u, v \notin N(u) \cup N(v)$ and it follows that $|N(u) \cup N(v)| \leq |V(G) - \{u, v\}| \leq n - 2$, a contradiction. So

$$\delta(G) \geq 3. \quad (2)$$

Lemma 2.1 *If $\delta(G) = 3$ and $a, b \in V(G)$ with $d(a, b) = 2$, then for any $x \in V(G) - \{a, b\}$, $x \in N(a) \cup N(b)$.*

Proof If $\exists x \in V(G) - \{a, b\}$ such that $x \notin N(a) \cup N(b)$, then $NC_2 \leq |N(a) \cup N(b)| \leq |V(G)| - |\{x\}| - |\{a, b\}| \leq n - 3 = n - \delta$, contrary to (1). □

Lemma 2.2 *Let $x, y \in V(G)$ and $P_m = x_0 x_1 \cdots x_m$ be an (x, y)-path of length m with $x = x_0$ and $y = x_m$. Then each of following holds.*

(i) *If P_m is a shortest (x, y)-path, then $m \leq 4$;*
(ii) *If P_m is a shortest (x, y)-path with $d_G(x, y) \geq 2$, then G also has an (x, y)-path of length $m + 1$.*
(iii) *If $d_G(x, y) = 1$ and P_m is a shortest (x, y)-path in $G - xy$, then $m \leq 4$;*
(iv) *If $d_G(x, y) = 1$ and P_m is a shortest (x, y)-path in $G - xy$ with $m \geq 3$, then $G - xy$ also has an (x, y)-path of length $m + 1$ and so does G.*

Proof (i) By way of contradiction we assume that $m \geq 5$. Since P_m is a shortest (x, y)-path in G with $m \geq 5$, $d(x_0, x_2) = 2$ and $N_{P_m}(x_m) = \{x_{m-1}\}$, $x_{m-1}x_0$, $x_{m-1}x_2 \notin E(G)$. If $N_{G-V(P_m)}(x_m) \cap (N_G(x_0) \cup N_G(x_2)) = \emptyset$, then $|N_G(x_0) \cup N_G(x_2)| \leq |V(G)| - |N_{G-V(P_m)}(x_m) \cup \{x_{m-1}\}| = |V(G)| - |N_G(x_m)| = n - \delta(G)$, a contradiction. So $\exists u \in N_{G-V(P_m)}(x_m)$ such that $u \in N(x_0) \cup N(x_2)$. Then either $x_0 u x_m$ is an (x, y)-path of length 2 or $x_0 x_1 x_2 u x_m$ is an (x, y)-path of length 4 in G, which contradicts that P_m is a shortest (x, y)-path with $m \geq 5$.

(ii) Since $d(x, y) \geq 2$ and P_m is a shortest (x, y)-path, $d(x_0, x_2) = 2$ and $N_{P_m}(x_1) = \{x_0, x_2\}$. Then $\exists u \in N(x_1) - \{x_0, x_2\}$ such that $u \in N(x_0) \cup N(x_2)$ otherwise $|N(x_0) \cup N(x_2)| \leq |V(G)| - |N(x_1)| \leq n - \delta(G)$, a contradiction. Then $x_0 u x_1 x_2 \cdots x_m$ or $x_0 x_1 u x_2 x_3 \cdots x_m$ is an (x, y)-path of length $m + 1$.

(iii) By way of contradiction we assume that $m \geq 5$. Since P_m is a shortest (x, y)-path in $G - xy$ with $m \geq 5$, $d(x_0, x_2) = 2$ and $N_{P_m}(x_m) = \{x_{m-1}, x_0\}$, $x_{m-1}x_0, x_{m-1}x_2 \notin E(G)$. Then $\exists u \in N_{G-P_m}(x_m)$ such that $u \in N(x_0) \cup N(x_2)$ otherwise $|N(x_0) \cup N(x_2)| \leq |V(G)| - |N_{G-P_m}(x_m) \cup \{x_0, x_{m-1}\}| = n - |N(x_m)| \leq n - \delta(G)$, a contradiction. So $x_0 u x_m$ is an (x, y)-path of length 2 or $x_0 x_1 x_2 u x_m$ is an (x, y)-path of length 4 in $G - xy$, contrary to the fact that $x_0 x_1 \cdots x_m$ is a shortest (x, y)-path in $G - xy$ with $m \geq 5$.

(iv) Since $m \geq 3$ and P_m is a shortest (x, y)-path in $G - xy$, $d_G(x_0, x_2) = 2$ and $N_G(x_1) \cap V(P_m) = \{x_0, x_2\}$. Then $\exists u \in N(x_1) - \{x_0, x_2\}$ such that

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$u \in N(x_0) \cup N(x_2)$ otherwise $|N(x_0) \cup N(x_2)| \le |V(G)| - |N(x_1)| \le n - \delta(G)$, a contradiction. Then $x_0 u x_1 x_2 \cdots x_m$ or $x_0 x_1 u x_2 x_3 \cdots x_m$ is an (x, y)-path of length $m+1$ in $G - xy$. □

Lemma 2.3 *Let $x, y \in V(G)$, $P_m = x_0 x_1 \cdots x_m$ be an (x, y)-path of length m and and for some i with $0 \le i < m$, $\exists u \in N_{G-P_m}(x_i)$, $v \in N_{G-P_m}(x_{i+1})$ with $u \ne v$ for $x_i, x_{i+1} \in V(P_m)$. If G does not have an (x, y)-path of length $m+2$, then $uv \notin E(G)$.*

Proof If $uv \in E(G)$, then $x_0 x_1 \cdots x_i u v x_{i+1} \cdots x_m$ is an (x, y)-path of length $m+2$, a contradiction. □

3 Base case

Theorem 3.1 *For any pair of distinct vertices $x, y \in V(G)$, one of the following holds.*

(i) *$G \in \{G_1\}$ (see Fig. 1) and G has (x, y)-paths of length of* 5 *and* 6;
(ii) *$G \notin \{G_1\}$ and $\exists k \in \{2, 3, 4\}$ such that G has (x, y)-paths of length k and $k+1$.*

Proof By Lemma 2.2(i), $\exists$ a shortest (x, y)-path of length ≤ 4. If $d_G(x, y) = 2, 3$ or 4, by Lemma 2.2(ii), G has an (x, y)-path of length 3, 4, 5 respectively, done. Next we assume that $d_G(x, y) = 1$. Let P_m be a shortest (x, y)-path in $G - xy$. By Lemma 2.2(iii) and (iv) if $d_{G-xy}(x, y) = 3$ or 4, then G has an (x, y)-path of length 4, 5 respectively, done. So we assume that $d_{G-xy}(x, y) = 2$. Let $x_0 x_1 x_2 = P_2$ be a shortest (x, y)-path of length 2 in $G - xy$. Since $d_G(x, y) = 1$, $x_0 x_2 \in E(G)$. By way of contradiction, we assume that

$$G \text{ does not have an } (x, y)\text{-path of length 3.} \tag{3}$$

Since $\delta(G) \ge 3$, $N_{G-P_2}(x_0) \ne \emptyset$ and $N_{G-P_2}(x_2) \ne \emptyset$.

Case 1 $\exists u \in N_{G-P_2}(x_0)$ but $u \notin N_{G-P_2}(x_2)$. Since $x_0 x_2 \in E(G)$, $d_G(u, x_2) = 2$. By (3) $x_1 u \notin E(G)$. Then $\exists v \in N(x_1) - \{x_0, x_2\}$ such that $u \ne v \in N(u) \cup N(x_2)$ otherwise $|N(u) \cup N(x_2)| \le |V(G)| - |N(x_1) - \{x_0\} \cup \{u\}| \le n - \delta(G)$, a contradiction. By (3) $v x_2 \notin E(G)$. So $vu \in E(G)$ and $x_0 u v x_1 x_2$ is an (x, y)-path of length 4. Since $u x_1 \notin E(G)$, $u x_2 \notin E(G)$ and $\delta(G) \ge 3$, $N_{G-P_2-v}(u) \ne \emptyset$. Since $d(v, x_2) = 2$ and $u x_2 \notin E(G)$, then $\exists u_1 \in N_G(u) - \{x_0, v, x_2\}$ such that $u_1 \in N(v) \cup N(x_2)$ otherwise $|N(v) \cup N(x_2)| \le |V(G)| - |N(u) - \{x_0\} \cup \{x_2\}| \le n - \delta(G)$, a contradiction. If $u_1 x_2 \in E(G)$, $x_0 u u_1 x_2$ is an (x, y)-path of length 3, contrary to (3). If $u_1 v \in E(G)$, $x_0 u u_1 v x_1 x_2$ is an (x, y)-path of length 5 and so G has an (x, y)-path of length 4 and 5, done.

Case 2 $N_{G-P_2}(x_0) \subseteq N(x_2)$. By symmetry, $N_{G-P_2}(x_0) = N_{G-P_2}(x_2)$.

If $N_{G-P_2}(x_0)$ has two vertices (say z_1, z_2) adjacent to each other, then by $N_{G-P_2}(x_0) = N_{G-P_2}(x_2)$, $x_0 z_1 z_2 x_2$ is an (x_0, x_2)-path of length 3, contrary to (3). Thus

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$$N_{G-P_2}(x_0) = N_{G-P_2}(x_2) \text{ is an independent set.} \tag{4}$$

For any $v \in N_{G-P_2}(x_1)$, if $v \in N_G(x_0) \cup N_G(x_2)$, then $x_0 v x_1 x_2$ or $x_0 x_1 v x_2$ is an (x_0, x_2)-path of length 3, contrary to (3). So

$$N_{G-P_2}(x_1) \cap (N_{G-P_2}(x_0) \cup N_{G-P_2}(x_2)) = \emptyset. \tag{5}$$

Subcase 2.1 $\delta(G) \geq 4$. Then $|N_{G-P_2}(x_0)| \geq 2$. By $N_{G-P_2}(x_0) = N_{G-P_2}(x_2)$, let $u_1, u_2 \in N_{G-P_2}(x_0) = N_{G-P_2}(x_2)$. By (5), $u_1, u_2 \notin N_{G-P_2}(x_1)$. By (4), $d(u_1, u_2) = 2$. If $N_{G-P_2}(x_1) \cap (N_{G-P_2}(u_1) \cup N_{G-P_2}(u_2)) = \emptyset$, then $|N_G(u_1) \cup N_G(u_2)| \leq |V(G)| - |N[x_1] - \{x_0, x_2\} \cup \{u_1\}| \leq n - \delta$, a contradiction. So $\exists v_1 \in N_{G-P_2}(x_1)$ such that $v_1 u_1 \in E(G)$ or $v_1 u_2 \in E(G)$. Without loss of generality we assume that $v_1 u_1 \in E(G)$. Then $x_0 u_1 v_1 x_1 x_2$ is an (x, y)-path of length 4. By (3) $v_1 x_0 \notin E(G)$, $v_1 x_2 \notin E(G)$. As $\delta(G) \geq 4$, $N_G(v_1) - V(P_2) - \{u_1, u_2\} \neq \emptyset$. Since $d(u_1, u_2) = 2$, $\exists v_1' \in N_G(v_1) - \{x_1, u_1, u_2\}$ such that either $v_1' u_1 \in E(G)$ or $v_1' u_2 \in E(G)$ otherwise $|N_G(u_1) \cup N_G(u_2)| \leq |V(G)| - |N(v_1)| \leq n - \delta(G)$, a contradiction. Then $x_0 u_1 v_1' v_1 x_1 x_2$ or $x_0 u_2 v_1' v_1 x_1 x_2$ is an (x, y)-path of length 5, respectively. Hence G has an (x, y)-path of length 4 and 5, done.

Subcase 2.2 $\delta(G) = 3$. If $|N_{G-P_2}(x_0)| \geq 2$, let $u_1, u_2 \in N_{G-P_2}(x_0)$. By (4), $d(u_1, u_2) = 2$. By Lemma 2.1, $x_1 \in N(u_1) \cup N(u_2)$, then $x_0 u_1 x_1 x_2$ or $x_0 u_2 x_1 x_2$ is an (x, y)-path of length 3, contrary to (3). So $|N_{G-P_2}(x_0)| = |N_{G-P_2}(x_2)| = 1$. We assume that

$$N_{G-P_2}(x_0) = N_{G-P_2}(x_2) = \{y_0\}. \tag{6}$$

Next we show that $V(G) - V(P_2) - \{y_0\}$ induces a complete graph. Let $G_1, \ldots, G_t$ be components of $G - V(P_2) - \{y_0\}$. If $\exists u_1, u_2 \in V(G_i)$ such that $d(u_1, u_2) = 2$, then by Lemma 2.1 $x_0 \in N(u_1) \cup N(u_2)$, contrary to (6). So each component G_i is complete. If $t \geq 2$, since $\kappa(G) \geq 2$, by (6) each component has at least two vertices adjacent to x_1 and to y_0 respectively. Then $\exists w_1 \in V(G_i)$, $w_2 \in V(G_j)$ such that $w_1 y_0 \in E(G)$, $w_2 y_0 \in E(G)$ and so $d(w_1, w_2) = 2$. By Lemma 2.1 $x_0 \in N(w_1) \cup N(w_2)$, contrary to (6). Hence $V(G) - V(P_2) - \{y_0\}$ induces a complete graph, denoted by $G[V(K_t)]$.

Since $n \geq 7$, $|V(G) - V(P_2) - \{y_0\}| = |V(K_t)| \geq 3$. Since $\kappa(G) \geq 2$, by Menger's Theorem, $\exists w_1, u_2 \in V(G) - V(P_2) - \{y_0\}$ such that $w_1 x_1 \in E(G)$, $w_2 y_0 \in E(G)$. Then $x_0 y_0 w_2 w_1 x_1 x_2$ is an (x, y)-path of length 5. If $\exists u' \in V(K_t)$ with $u' y_0, u' x_1 \in E(G)$, then $x_0 y_0 u' x_1 x_2$ is an (x, y)-path of length 4. Hence G has an (x, y)-path of length 4 and 5, done. So for any $u' \in V(K_t)$, u' cannot be adjacent to both y_0 and x_1. By (3), $y_0 x_1 \notin E(G)$ and $d(y_0, x_1) = 2$. By Lemma 2.1, for any $z \in V(K_t)$, $z \in N(y_0) \cup N(x_1)$. Therefore this is the class G_3 of graphs depicted as in Fig. 1. Let $u_3 \in V(K_t) - \{w_1, w_2\}$. Then $x_0 y_0 w_2 w_1 x_1 x_2$ and $x_0 y_0 w_2 u_3 w_1 x_1 x_2$ are (x, y)-path of length 5 and 6, respectively. □

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4 Proof of Theorems 1.3 and 1.4 (Induction)

Lemma 4.1 *Let P_m be an (x, y)-path of length m and $u \in V(G) - V(P_m)$ with $|N^+_{P_m}(u)| \geq 2$. If G does not have an (x, y)-path of length $m+2$, then one of the following must hold.*

(i) *$\exists$ a pair $x_{i+1}, x_{j+1} \in N^+_{P_m}(u)$ such that $x_{i+1}x_{j+1} \in E(G)$;*

(ii) *for every pair of $x_{k+1}, x_{h+1} \in N^+_{P_m}(u)$ (where $k < h$) with $\{x_{k+1}, x_{k+2}, \ldots, x_{h-1}\} \cap N_{P_m}(u) = \emptyset$, $\exists r, s, t$ such that one of the following holds*

$$\begin{cases} x_r x_{k+1}, x_{h+1}x_{r+1} \in E(G) & : \quad 1 \leq r < k \\ x_{s+1}x_{k+1}, x_{h+1}x_s \in E(G) & : \quad k+1 < s < h \\ x_t x_{k+1}, x_{h+1}x_{t+1} \in E(G) & : \quad h+1 < t < m \end{cases}$$

Proof We assume that (i) fails to prove (ii). By contradiction, assume further that no such r, s or t can be found. Since (i) does not hold, $x_{k+1} \neq x_h$. And as $\{x_{k+1}, x_{k+2}, \ldots, x_{h-1}\} \cap N_{P_m}(u) = \emptyset$, $d(u, x_{k+1}) = 2$. By Lemma 2.3, $N_{G-P_m}(x_{h+1}) \cap N_{G-P_m}(u) = \emptyset$. If $\exists w \in N_{G-P_m}(x_{h+1})$ such that $wx_{k+1} \in E(G)$, then $x_0 \cdots x_k u x_h x_{h-1} \cdots x_{k+1} w x_{h+1} \cdots x_m$ is an (x, y)-path of length $m+2$, contrary to the assumption. So $N_{G-P_m}(x_{h+1}) \cap N_{G-P_m}(x_{k+1}) = \emptyset$. Let $T_1 = x_0x_1 \cdots x_k$, $T_2 = x_{k+1}x_{k+2} \cdots x_h$ and $T_3 = x_{h+1}x_{h+2} \cdots x_m$. Since $\{x_{k+1}, x_{k+2}, \ldots, x_{h-1}\} \cap N_{P_m}(u) = \emptyset$ and (i), (ii) do not hold, for any $z \in N_G(x_{k+1}) \cup N_G(u)$,

$$z \notin N^-_{T_1 - \{x_0\}}(x_{h+1}) \cup N^+_{T_2 - \{x_h\}}(x_{h+1}) \cup N^-_{T_3}(x_{h+1}).$$

and $N^-_{T_1-\{x_0\}}(x_{h+1})$, $N^+_{T_2-\{x_h\}}(x_{h+1})$ and $N^-_{T_3}(x_{h+1})$ are pairwise disjoint. Then $|N_G(x_{k+1}) \cup N_G(u)| \leq |V(G)| - (|N_{G-P_m}(x_{h+1})| + |N^-_{T_1-\{x_0\}}(x_{h+1}) \cup N^+_{T_2-\{x_h\}}(x_{h+1}) \cup N^-_{T_3}(x_{h+1}) \cup \{u, x_{k+1}\} - \{x_0, x_h\}|) = |V(G)| - |N_{G-P_m}(x_{h+1}) \cup N_{P_m}(x_{h+1})| \leq n - \delta(G)$, contrary to (1). □

Corollary 4.2 *Let P_m be an (x, y)-path of length m and $u \in V(G) - V(P_m)$ with $|N^+_{P_m}(u)| \geq 2$. If G does not have an (x, y)-path of length $m+2$, then G has an (x, y)-path P_{m+1} of length $m+1$ with $V(P_{m+1}) = V(P_m) \cup \{u\}$.*

Proof If Lemma 4.1(i) holds, then $\exists x_{k+1}, x_{h+1} \in N^+_{P_m}(u)$ with $x_{k+1}x_{h+1} \in E(G)$ $(k < h < m)$. Hence $x_0x_1 \cdots x_k u x_h x_{h-1} \cdots x_{k+1} x_{h+1} \cdots x_m$ is an (x_0, x_m)-path of length $m+1$. Next we assume that Lemma 4.1(ii) holds. If $x_r x_{k+1}, x_{h+1}x_{r+1} \in E(G)$, then $x_0x_1 \cdots x_r x_{k+1} x_{k+2} \cdots x_h u x_k x_{k-1} \cdots x_{r+1} x_{h+1} x_{h+2} \cdots x_m$ is an (x_0, x_m)-path of length $m+1$. If $x_{s+1}x_{k+1}, x_{h+1}x_s \in E(G)$, then $x_0x_1 \cdots x_k u x_h x_{h-1} \cdots x_{s+1} x_{k+1} x_{k+2} \cdots x_s x_{h+1} x_{h+2} \cdots x_m$ is an (x_0, x_m)-path of length $m+1$. If $x_t x_{k+1}, x_{h+1}x_{t+1} \in E(G)$, then $x_0x_1 \cdots x_k u x_h x_{h-1} \cdots x_{k+1} x_t x_{t-1} \cdots x_{h+1} x_{t+1} x_{t+2} \cdots x_m$ is an (x_0, x_m)-path of length $m+1$. □

Lemma 4.3 *Let $P_m = x_0x_1x_2 \cdots x_m$ be an (x, y)-path of length m in G. If $\exists w, w' \in V(G) - V(P_m)$ satisfying both of the following,*

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(i) *both* $|N_{P_m}(w)| \geq 2$ *and* $|N_{P_m}(w')| \geq 2$, *and*

(ii) *both* $N_{P_m}(w) - \{x_0, x_m\} \neq \emptyset$ *and* $N_{P_m}(w') - \{x_0, x_m\} \neq \emptyset$, *then* G *has an* (x, y)*-path of length* $m + 2$.

Proof By way of contradiction, we assume that G does not have an (x, y)-path of length $m + 2$. If $|N^+_{P_m}(w)| = |N^+_{P_m}(w')| = 1$, then $x_m \in N_{P_m}(w)$, $x_m \in N_{P_m}(w')$. Reverse the order of P_m to get P'_m, then by (i) and (ii) $|N^+_{P'_m}(w)| \geq 2$, $|N^+_{P'_m}(w')| \geq 2$. So we may assume that $\{x_i, x_j\} \subseteq N_{P_m}(w)$ with $0 \leq i < j < m$. By Corollary 4.2, G has an (x_0, x_m)-path P_{m+1} with $V(P_{m+1}) = V(P'_m) \cup \{w\}$.

Note that $N_{P_m}(w') \subseteq V(P_m) \subseteq V(P_{m+1})$. Thus $|N_{P_{m+1}}(w')| \geq 2$. If $x_m \notin N_{P_{m+1}}(w')$ or if $|N_{P_{m+1}}(w')| \geq 3$, then $|N^+_{P_{m+1}}(w')| \geq 2$, and we can apply Corollary 4.2 to P_{m+1} and w' to find an (x_0, x_{m+2})-path P_{m+2} with $V(P_{m+2}) = V(P_{m+1}) \cup \{w'\}$. Therefore, we may assume that $N_{P_{m+1}}(w') = N_{P_m}(w') = \{x_l, x_m\}$, with $0 < l < m$. Reverse the order of P_{m+1} to get an (x_{m+1}, x_0)-path Q_{m+1}. Then $|N^+_{Q_{m+1}}(w')| \geq 2$, and so we can apply Corollary 4.2 to Q_{m+1} and w' to find an (x_{m+2}, x_0)-path Q_{m+2} with $V(Q_{m+2}) = V(Q_{m+1}) \cup \{w'\}$. Therefore, in any case, we can find an (x_0, x_m)-path of length $m + 2$, a contradiction. □

Theorem 4.4 *Let* $x, y \in V(G)$. *If* G *has an* (x, y)*-path* $P_2 = x_0x_1x_2$ *of length* 2, *then either* $G \in \{G_1, G_2, G_4\}$ *(see Figs.* 1, 2, 4*) or* G *has an* (x, y)*-path of length* 4.

Proof By way of contradiction we assume that

$$G \text{ does not have an } (x, y)\text{-path of length } 4. \tag{7}$$

Case 1 $\delta(G) = 3$. Then $N(x_0) - \{x_1, x_2\} \neq \emptyset$ and $N(x_2) - \{x_0, x_1\} \neq \emptyset$.

Subcase 1.1 $|N_{G-P_2}(x_0) \cup N_{G-P_2}(x_2)| \geq 2$. Then $\exists y_0, y_2 \in V(G) - V(P_2)$ with $y_0 \neq y_2$ such that $x_0y_0, x_2y_2 \in E(G)$. First we assume that $y_0y_2 \in E(G)$.

By Lemma 2.3 for each $i \in \{0, 2\}$, $y_ix_1 \notin E(G)$ and so $d(y_i, x_1) = 2$. Then by Lemma 2.1, $N_G(y_i) \cup N_G(x_1) = V(G) - \{y_i, x_1\}$. If $\exists u \in V(G) - (V(P_2) \cup \{y_0, y_2\})$ such that $uy_0 \in E(G)$, then by Lemma 2.3, $ux_1 \notin E(G)$. Since $d(y_2, x_1) = 2$, by Lemma 2.1 $uy_2 \in E(G)$, then $x_0y_0uy_2x_2$ is an (x, y)-path of length 4, contrary to (7). So by symmetry

$$\text{for any } u \in V(G) - (V(P_2) \cup \{y_0, y_2\}), uy_0, uy_2 \notin E(G). \tag{8}$$

Since $d(y_0, x_1) = 2$, by Lemma 2.1, for any $u \in V(G) - (V(P_2) \cup \{y_0, y_2\})$, $ux_1 \in E(G)$. Therefore $V(G) - (V(P_2) \cup \{y_0, y_2\}) \subseteq N_G(x_1)$.

Since $n \geq 7$, $|V(G) - (V(P_2) \cup \{y_0, y_2\})| \geq 2$. If there exist two vertices $w_1, w_2 \in V(G) - (V(P_2) \cup \{y_0, y_2\})$ such that $d(w_1, w_2) = 2$, then by Lemma 2.1, we must have $y_0 \in N_G(w_1) \cup N_G(w_2)$, contrary to (8). It follows that $V(G) - (V(P_2) \cup \{y_0, y_2\})$ induces a complete subgraph $K_t \cong K_{n-5}$, where $n - 5 \geq 7 - 5 = 2$. Since G is 2-connected, x_1 is not a cut vertex of G, and also $N_{G-P_m}(x_1) \cap (N_{G-P_m}(y_0) \cup N_{G-P_m}(y_2)) = \emptyset$ by (8), we can find $u_1 \in V(G) - (V(P_2) \cup \{y_0, y_2\})$ such that $u_1x_0 \in E(G)$ (or respectively, $u_1x_2 \in E(G)$). Since $|V(G) - (V(P_2) \cup \{y_0, y_2\})| \geq 2$,

$\exists u_2 \in V(G)-(V(P_2)\cup\{y_0, y_2, u_1\})$. Hence $x_0u_1u_2x_1x_2$ (or respectively, $x_0x_1u_2u_1x_2$) is an (x_0, x_2)-path of length 4, contrary to (7).

Next we assume $y_0y_2 \notin E(G)$. By (7), at most one edge in $\{y_0x_1, y_2x_1\}$ is in $E(G)$ and we assume that $y_2x_1 \notin E(G)$. So $d(y_2, x_1) = 2$ and by Lemma 2.1 and $y_0y_2 \notin E(G)$, $y_0x_1 \in E(G)$. If $\exists u' \in N_G(y_0) - (V(P_2) \cup \{y_0, y_2\})$, then by Lemma 2.1, $u' \in N_G(x_1) \cup N_G(y_2)$. Each case is contrary to Lemma 7. Thus

$$N_G(y_0) \subseteq V(P_2). \tag{9}$$

Since $\delta(G) = 3$, $y_0x_2 \in E(G)$. Then $d(y_0, y_2) = 2$. As $y_0y_2 \notin E(G)$, $d_G(y_0, y_2) = 2$. By (9) and Lemma 2.1, $V(G) - (V(P_2) \cup \{y_0, y_2\}) \subseteq N_G(y_2)$. Since $n \geq 7$, $|V(G) - (V(P_2) \cup \{y_0, y_2\})| \geq 2$. Let $u_1, u_2 \in V(G) - (V(P_2) \cup \{y_0, y_2\})$, if $u_1u_2 \notin E(G)$, then by Lemma 2.1, $x_1 \in N_G(u_1) \cup N_G(u_2)$, contrary to Lemma 2.3. Hence $V(G) - (V(P_2) \cup \{y_0, y_2\})$ induces a complete subgraph $K_t \cong K_{n-5}$ in G. Since $\kappa(G) \geq 2$ and $N_G(y_0) \subseteq V(P_2)$, we may assume that $u_1 \in N_G(x_0)$ and $u_2 \in N_G(y_2)$. It follows that $x_0u_1u_2y_2x_2$ is an (x_0, x_2)-path of length 4, contrary to (7). Therefore, Case 1.1 is precluded.

Subcase 1.2 $|N_{G-P_2}(x_0) \cup N_{G-P_2}(x_2)| = 1$. Let $y_0 \in N_{G-P_2}(x_0) \cup N_{G-P_2}(x_2)$.

Since $\delta(G) = 3$ and $N_{G-P_2}(x_0) \cup N_{G-P_2}(x_2) = \{y_0\}$, we must have $x_0x_2 \in E(G)$ and for any $y \in V(G) - V(P_2) - y_0$, $yx_0, yx_2 \notin E(G)$. Then

$$N_G(x_0) - \{x_2\} = N_G(x_2) - \{x_0\} = \{x_1, y_0\}. \tag{10}$$

Since $\kappa(G) \geq 2$, if $G - V(P_2) - \{y_0\}$ is not connected, then by (10) each component is adjacent to both y_0 and x_1. So $\exists u, v$ from two different components such that $uy_0, vy_0 \in E(G)$ and thus $d(u, v) = 2$. So $|N_G(u) \cup N_G(v)| \leq n - |\{x_0, x_2, u\}| = n - 3 = n - \delta(G)$, a contradiction. Similarly we can prove that $V(G) - V(P_2) - \{y_0\}$ induces a complete subgraph K_t of G. If $\exists u' \in V(K_t)$ with $u'y_0, u'x_1 \in E(G)$, then $x_0y_0u'x_1x_2$ is an (x, y)-path of length 4, contrary to (7). So for any $u' \in V(K_t)$, if $u'y_0 \in E(G)$, then $u'x_1 \notin E(G)$ and if $u'x_1 \in E(G)$, then $u'y_0 \notin E(G)$. If $y_0x_1 \notin E(G)$, then $d(y_0, x_1) = 2$. By Lemma 2.1, for any $w \in V(G) - V(P_2) - \{y_0\}$, exactly one of $wy_0 \in E(G)$ and $wx_1 \in E(G)$ holds. If $y_0x_1 \in E(G)$, then for any $w \in V(G) - V(P_2) - \{y_0\}$, w is not adjacent to both y_0 and x_1. This class G_1 of graphs is depicted in Fig. 1.

Case 2 $\delta(G) \geq 4$.

Subcase 2.1 $|N_{G-P_2}(x_0) \cap N_{G-P_2}(x_1)| \geq 1$ or $|N_{G-P_2}(x_2) \cap N_{G-P_2}(x_1)| \geq 1$. We may assume that $y_0^1 \in N_{G-P_2}(x_0) \cap N_{G-P_2}(x_1)$. Since $\delta(G) \geq 4$, $\exists y_1 \in N_{G-P_2-y_0^1}(x_1)$. By Lemma 2.3 $y_0^1y_1 \notin E(G)$. By $\delta(G) \geq 4$, $\exists z_0 \in N_{G-P_2-\{y_1\}}(y_0^1)$. By (7), $z_0x_1, z_0x_2 \notin E(G)$ and $N_{G-P_2-y_0^1}(x_2) \cap (N_{G-P_2-y_0^1}(z_0) \cup N_{G-P_2-y_0^1}(x_1)) = \emptyset$. We have the following observations.

(A) $y_0^1x_2 \in E(G)$ and $x_0x_2 \in E(G)$. Otherwise if $y_0^1x_2 \notin E(G)$, then $|N(z_0) \cup N(x_1)| \leq |V(G)| - |N(x_2) - \{x_0\} \cup z_0| = n - \delta(G)$, a contradiction; if $x_0x_2 \notin$

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$E(G)$, then $|N(z_0) \cup N(x_1)| \le |V(G)| - |N(x_2) - \{y_0^1\} \cup z_0| = n - \delta(G)$, a contradiction.

(B) Let $y_2 \in N_{G-P_2-\{y_0^1, y_1\}}(x_2)$. Then $x_0 y_2 \in E(G)$ and $N(x_0) = \{x_1, x_2, y_0^1, y_2\}$. So $\delta(G) = 4$.

If $\exists y_0 \in N(x_0) - V(P_2) - \{y_0^1, y_1, y_2\}$, then $y_0 y_0^1 \notin E(G)$ otherwise $x_0 y_0 y_0^1 x_1 x_2$ is an (x, y)-path of length 4, contrary to (7). So $|N(y_0^1) \cup N(y_0)| \le |V(G)| - |N(y_2) - \{x_0, x_2\} \cup \{y_0^1, y_1\}| = n - \delta(G)$, contrary to (1). Since $\delta(G) \ge 4$, $y_0^1 y_2 \in E(G)$ and so $N(x_0) = \{x_1, x_2, y_0^1, y_2\}$.

(C) $d(x_1) = d(x_2) = 4$, and so $N(x_1) = \{x_0, x_2, y_0^1, y_1\}$ and $N(x_2) = \{x_0, x_1, y_0^1, y_2\}$.

By Lemma 2.3, $N_{G-P_2}(x_1) \cap (N(y_0^1) \cup N(y_2)) = \emptyset$. If $|N(x_1)| \ge 5$, then $|N(y_0^1) \cup N(y_2)| \le |V(G)| - |N(x_1) - \{x_0, x_2\} \cup y_2| \le n - 4 = n - \delta(G)$, contrary to (1). Similarly, if $|N(x_2)| \ge 5$, $|N(y_0^1) \cup N(y_1)| \le |V(G)| - |N(x_2) - \{x_0, x_1\} \cup y_1| \le n - 4 = n - \delta(G)$, a contradiction.

(D) $N(y_0^1) = \{x_0, x_1, x_2, z_0\}$.

If $|N(y_0^1)| \ge 5$, then $|N(x_1) \cup N(y_2)| \le |V(G)| - |N(y_0^1) - \{x_0, x_2\} \cup y_2| \le n - 4 = n - \delta(G)$, a contradiction.

(E) $z_0 y_1 \in E(G)$ and $z_0 y_2 \notin E(G)$. So $N_{G[P_2 \cup \{y_0^1, y_1, y_2\}]}(z_0) = \{y_0^1, y_1\}$.

By (D), if $z_0 y_1 \notin E(G)$, then $|N(x_0) \cup N(y_1)| \le |V(G)| - |N(y_0^1) - \{x_1, x_2\} \cup \{z_0, y_1\}| = n - 4$, a contradiction. If $z_0 y_2 \in E(G)$, then $x_0 y_0^1 z_0 y_2$ is an (x, y)-path of length 4, contrary to (7). By (B) and (C), $N_{G[P_2 \cup \{y_0^1, y_1, y_2\}]}(z_0) = \{y_0^1, y_1\}$.

(F) For any $v \in V(G) - V(P_2) - \{y_0^1, y_1, y_2, z_0\}$, $vz_0, vy_1, vy_2 \in E(G)$.

If $\exists v \in V(G) - V(P_2) - \{y_0^1, y_1, y_2, z_0\}$ such that $vy_2 \notin E(G)$, $|N(x_1) \cup N(y_2)| \le |V(G)| - |\{z_0, x_1, y_2, v\}| = n - 4$, a contradiction; if $vy_1 \notin E(G)$, $|N(y_0^1) \cup N(y_1)| \le |V(G)| - |\{y_0^1, y_1, y_2, v\}| = n - 4$, a contradiction; if $vz_0 \notin E(G)$, $|N(z_0) \cup N(x_1)| \le |V(G)| - |\{z_0, x_1, y_2, v\}| = n - 4$, a contradiction.

(G) For any $v_1, v_2 \in V(G) - V(P_2) - \{y_0, y_1, y_2, z_0\}$, $v_1 v_2 \in E(G)$.

If $\exists v_1, v_2 \in V(G) - V(P_2) - \{y_0, y_1, y_2, z_0\}$ such that $v_1 v_2 \notin E(G)$, then by (F), $d(v_1, v_2) = 2$. By (B), (C) and (D), $(y_0^1 \cup V(P_2)) \cap (N(v_1) \cup N(v_2)) = \emptyset$, then $|N(v_1) \cup N(v_2)| \le |V(G)| - |y_0^1 \cup V(P_2)| = n - 4 = n - \delta(G)$, contrary to (1).

By combining (A)–(G), we conclude that $G \in \{G_4\}$.

Subcase 2.2 $|N_{G-P_2}(x_0) \cap N_{G-P_2}(x_1)| = 0$ and $|N_{G-P_2}(x_2) \cap N_{G-P_2}(x_1)| = 0$. Then by symmetry for any $y_1 \in N_{G-P_2}(x_1)$, $y_1 x_0 \notin E(G)$ and $y_1 x_2 \notin E(G)$.

First we show that $N_{G-P_2}(x_0)$ is complete. If $\exists y_0^1, y_0^2 \in N_{G-P_2}(x_0)$ such that $y_0^1 y_0^2 \notin E(G)$, then $d(y_0^1, y_0^2) = 2$. By Lemma 2.3 $N_{G-P_2}(x_1) \cap (N_{G-P_2}(y_0^1) \cup N_{G-P_2}(y_0^2)) = \emptyset$, then $|N_G(y_0^1) \cup N_G(y_0^2)| \le |V(G)| - |N_G(x_1) - \{x_0, x_2\} \cup \{y_0^1, y_0^2\}| \le n - \delta(G)$, a contradiction. So $N_{G-P_2}(x_0)$ is complete. Next we show

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$N_{G-P_2}(x_0) = N_{G-P_2}(x_2)$. If $\exists y_0^1 \in N_{G-P_2}(x_0)$ such that $y_0^1 x_2 \notin E(G)$, then $d(y_0^1, x_2) = 2$. Since $N_{G-P_2}(x_1) \cap N_{G-P_2}(x_2) = \emptyset$, and by Lemma 2.3, $N_{G-P_2}(x_1) \cap N_{G-P_2}(y_0^1) = \emptyset$, we have $|N_G(y_0^1) \cup N_G(x_2)| \le |V(G)| - |N_G(x_1) - \{x_0\} \cup \{y_0^1\}| \le n - \delta(G)$, a contradiction. By symmetry,

$$N_{G-P_2}(x_0) = N_{G-P_2}(x_2) \text{ is complete.} \tag{11}$$

If $x_0 x_2 \notin E(G)$, then $d(x_0, x_2) = 2$. By Subcase 2.2 assumption that $N_{G-P_2}(x_1) \cap (N_{G-P_2}(x_0) \cup N_{G-P_2}(x_2)) = \emptyset$, so $|N_G(x_0) \cup N_G(x_2)| \le |V(G)| - |N_G(x_1)| \le n - \delta(G)$, a contradiction. So $x_0 x_2 \in E(G)$.

If $|N_{G-P_2}(x_0)| \ge 3$, let $u_1, u_2, u_3 \in N_{G-P_2}(x_0)$. By (11), $x_0 u_1 u_2 u_3 x_2$ is an (x, y)-path of length 4, contrary to (7). So we must have $|N_{G-P_2}(x_0)| = 2$ since $\delta(G) \ge 4$. Then $\delta(G) = 4$ and let $N_{G-P_2}(x_0) = \{u_1, u_2\}$. We show that $V(G) - V(P_2) - \{u_1, u_2\}$ induces a complete graph. If $\exists v_1, v_2 \in V(G) - V(P_2) - \{u_1, u_2\}$ such that $d(v_1, v_2) = 2$, then $x_0, x_2 \notin N_G(v_1) \cup N_G(v_2)$, contrary to (1). Let K_t denote the graph induced by $V(G) - V(P_2) - \{u_1, u_2\}$. By Lemma 2.3, $N_{K_t}(x_1) \cap (N_{K_t}(u_1) \cup N_{K_t}(u_2)) = \emptyset$. By (7) $N_{K_t}(u_1) \cap N_{K_t}(u_2) = \emptyset$. Since $d(x_1, u_1) = d(x_1, u_2) = 2$, $\delta(G) \ge 4$, $|N_{K_t}(u_1)| = |N_{K_t}(u_2)| = 1$. Thus the class of graphs is depicted in Fig. 2. Hence $G \in \{G_2\}$. □

Theorem 4.5 *Let $x, y \in V(G)$. If G has an (x, y)-path $P_m = x_0 x_1 \cdots x_m$ of length m with $3 \le m \le |V(G)| - 2$, then G has an (x, y)-path of length $m + 2$ or $G \in \{G_3\}$ (Fig. 3).*

Proof By way of contradiction we assume that

$$G \text{ does not have an } (x, y)\text{-path of length } m + 2. \tag{12}$$

By Lemma 4.3, we may assume that

$$|\{w \in V(G) - V(P_m) : |N_{P_m}(w)| \ge 2 \quad \text{and} \quad N_{P_m}(w) - \{x_0, x_m\} \ne \emptyset\}| \le 1. \tag{13}$$

Case 1 $\exists w \in V(G) - V(P_m)$ such that $w x_i \in E(G)$ for some $x_i \in V(P_m) - \{x_0, x_m\}$ and for any $v \in V(G) - V(P_m) - w$, $N_{P_m}(v) \subseteq \{x_0, x_m\}$.

Claim 1 (i) $G[V(G) - V(P_m) - w]$ is complete.
(ii) $G[V(P_m) - \{x_0, x_m\}]$ is complete.
(iii) $N_G(w) \subseteq V(P_m)$.
(iv) $G[V(P_m) - \{x_0, x_m\} \cup w]$ is complete.

Proof of Claim 1 (i) Let $G_1, \ldots, G_t$ be components of $G[V(G) - V(P_m) - w]$. First we show that each component G_i is complete. By way of contradiction that we assume that $\exists y_1, y_2 \in V(G_i)$ such that $d_{G_i}(y_1, y_2) = 2$. Since $m \ge 3$, $x_1 \in V(P_m)$ is an inner vertex. By Case 1 assumption, $N_G(x_1) \subseteq V(P_m) \cup w$. Then $|N_G(y_1) \cup N_G(y_2)| \le |V(G)| - |N_G[x_1] - \{x_0, x_m, w\} \cup \{y_1, y_2\}| \le n - \delta(G)$, a contradiction. Hence G_i is complete.

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By the assumption of Case 1, $N_{P_m \cup w}(G_i) \subseteq \{x_0, x_m, w\}$ for each $i \in \{1, 2, \ldots, t\}$. Since $\kappa(G) \geq 2$, $|N_{P_m}(G_i)| \geq 2$. If $t \geq 2$, then $\exists$ two vertices from distinct G_i and G_j respectively are adjacent to a same vertex in $\{x_0, x_m, w\}$. Assume that $\exists y_1' \in G_i, y_2' \in G_j$ such that $d_G(y_1', y_2') = 2$. Then $|N_G(y_1') \cup N_G(y_2')| \leq |V(G)| - |N_G[x_1] - \{x_0, x_m, w\} \cup \{y_1, y_2\}| \leq n - \delta(G)$, a contradiction. Hence $t = 1$. Thus $G[V(G) - V(P_m) - w]$ is complete.

(ii) By way of contradiction we suppose that $\exists x_l, x_k \in V(P_m) - \{x_0, x_m\}$ such that $d_G(x_l, x_k) = 2$. Since $|V(G) - V(P_m)| \geq 2$, let $y \in V(G) - V(P_m) - w$. By the assumption of Case 1, $N_{P_m \cup w}(y) \subseteq \{x_0, x_m, w\}$. Since x_l, x_k are both inner vertices, $|N_G(x_l) \cup N_G(x_k)| \leq |V(G)| - |N_G[y] - \{x_0, x_m, w\} \cup \{x_l, x_k\}| \leq n - \delta(G)$, a contradiction. Thus $G[V(P_m) - \{x_0, x_m\}]$ is complete.

(iiii) By way of contradiction we assume that w is adjacent to some vertex w_1 in $G[V(G) - V(P_m) - w]$. First we assume that $x_i \neq x_1$ and $x_i \neq x_{m-1}$. If $w_1 x_0 \in E(G)$ or $w_1 x_m \in E(G)$, then by Claim 1(ii), there is an (x_i, x_{m-1}) path T or (x_1, x_i) path T' of length $m - 2$ in $G[V(P_m) - \{x_0, x_m\}]$. And so $x_0 w_1 w x_i T x_{m-1} x_m$ or $x_0 x_1 T' x_i w w_1 x_m$ is an (x, y)-path of length $m + 2$, contrary to (12). Otherwise since $\kappa(G) \geq 2$, $\exists w_2 \in V(G) - V(P_m) - \{w, w_1\}$ such that either $w_2 x_0 \in E(G)$ or $w_2 x_m \in E(G)$. Similarly, if $w_2 x_0 \in E(G)$ or $w_2 x_m \in E(G)$, then by Claim 1(ii), there is an (x_i, x_{m-1}) path T or (x_1, x_i) path T' of length $m - 3$ in $G[V(P_m) - \{x_0, x_m\}]$. And so $x_0 w_2 w_1 w x_i T x_{m-1} x_m$ or $x_0 x_1 T' x_i w w_1 w_2 x_m$ is an (x, y)-path of length $m + 2$, contrary to (12).

Suppose that $x_i = x_1$. Then by Lemma 2.3, $x_0 w_1 \notin E(G)$. If $\exists w_2 \in V(G) - V(P_m) - \{w, w_1\}$ such that $w_2 x_0 \in E(G)$, then by Claim 1(i), $x_0 w_2 w_1 w x_1 x_3 \cdots x_m$ is an (x, y)-path of length $m+2$, contrary to (12). So $N_{G - V(P_m) - \{w\}}(w_1) \cap N_{G - V(P_m) - \{w\}}(x_0) = \emptyset$. If $x_0 x_{m-1} \notin E(G)$, then by Claim 1(ii), $x_1 x_{m-1} \in E(G)$ and so $d(x_0, x_{m-1}) = 2$. Together with the assumption of Case 1, $|N_G(x_0) \cup N_G(x_{m-1})| \leq |V(G)| - |N_G(w_1) - \{w\} \cup \{x_{m-1}\}| \leq n - \delta(G)$, contrary to (1). Hence $x_0 x_{m-1} \in E(G)$. If $w_1 x_m \in E(G)$, then $x_0 x_{m-1} x_{m-2} \cdots x_1 w w_1 x_m$ is an (x, y)-path of length $m+2$, contrary to (12). Otherwise since $\kappa(G) \geq 2$ and $N_{G - V(P_m) - \{w\}}(x_0) \cap (V(G) - V(P_m) - \{w\}) = \emptyset$, $\exists w_3 \in V(G) - V(P_m) - \{w, w_1\}$ such that $w_3 x_m \in E(G)$. Then $x_0 x_{m-1} x_{m-3} x_{m-4} \cdots x_1 w w_1 w_3 x_m$ $(m \geq 4)$ or $x_0 x_1 w w_1 w_3 x_m$ $(m = 3)$ is an (x, y)-path of length $m + 2$, contrary to (12). By symmetry the case $x_i = x_{m-1}$ can be excluded similarly as the case $x_i = x_1$.

(iv) By Claim 1(ii) it suffices to show that $w x_k \in E(G)$ for $k \in \{1, 2, \ldots, m - 1\}$. Assume that $x_{i-1} \in V(P_m) - \{x_0, x_m\}$ and $w x_{i-1} \notin E(G)$. Since $w x_i \in E(G)$, $d(x_{i-1}, w) = 2$. Let $y \in V(G) - V(P_m) - w$. By Claim 1(iii), $N_G(w) \subseteq V(P_m)$ and $N_{G - P_m}(y) \cap N_G[w] = \emptyset$. By the assumption of Case 1 $N_{P_m}(y) \subseteq \{x_0, x_m\}$. So $|N_G(x_{i-1}) \cup N_G(w)| \leq |V(G)| - |N_G[y] - \{x_0, x_m\} \cup \{w, x_{i-1}\}| \leq n - \delta(G)$, contrary to (1). Hence $w x_{i-1} \in E(G)$. Similarly $w x_{i-k} \in E(G)$ where $k \in \{2, \ldots, i - 1\}$ and $w x_{i+k} \in E(G)$ where $k \in \{1, 2, \ldots, m - i - 1\}$. So $G[V(P_m) - \{x_0, x_m\} \cup w]$ is complete. □

By Claim 1(iii), $N_G(w) \subseteq V(P_m)$. Since $\kappa(G) \geq 2$ and $\delta(G) \geq 3$, $|V(G) - V(P_m) - w)| \geq 2$ and $\exists v, v' \in V(G) - V(P_m) - w$ such that $v x_0 \in E(G)$ and

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$v'x_m \in E(G)$. By Claim 1(i), if $|V(G) - V(P_m) - w| \geq m + 1$, then there is a (v, v')-path P of length m. So $x_0 P x_m$ is an (x, y)-path of length $m + 2$, contrary to (12). Hence $2 \leq |V(G) - V(P_m) - w| \leq m$. By Claim 1(i), (iii) and (iv), this is the class of graphs depicted in Fig. 3 and so $G \in \{G_3\}$.

Case 2 For any $w \in V(G) - V(P_m)$, $N_{P_m}(w) \subseteq \{x_0, x_m\}$. The following claim can be proved by the argument similar to the Proof of Claim 1.

Claim 2 (i) $G[V(G) - V(P_m)]$ is complete.
(ii) $G[V(P_m) - \{x_0, x_m\}]$ is complete.

Since $\kappa(G) \geq 2$ and $\delta(G) \geq 3$, $\exists w, w' \in V(G) - V(P_m)$ such that $wx_0 \in E(G)$ and $w'x_m \in E(G)$. By Claim 2(i), if $|V(G) - V(P_m)| \geq m + 1$, then $G - V(P_m)$ is a (w, w')-path P of length m. So $x_0 P x_m$ is an (x, y)-path of length $m + 2$, contrary to (12). Hence $|V(G) - V(P_m)| \leq m$. By Claim 2(ii), this class of graphs is depicted in Fig. 3.

Case 3 $\exists w, w' \in V(G) - V(P_m)$ such that $wx_i \in E(G)$ and $w'x_j \in E(G)$ where x_i, x_j are inner vertices and $w \neq w'$. Since x_i, x_j are both inner vertices, by (13), one of $\{w, w'\}$ has only one neighbor in P_m. Without loss of generality we assume that

$$N_{P_m}(w) = \{x_i\} \text{ with } 1 \leq i \leq m - 1. \tag{14}$$

Claim 3 $x_{i-1}x_{i+k} \in E(G)$ for each k with $0 \leq k \leq m - i$ and $x_{i+1}x_{i-k} \in E(G)$ for each k with $0 \leq k \leq i$.

Proof of Claim 3 Clearly $x_{i-1}x_i \in E(G)$ and $x_{i+1}x_i \in E(G)$. First we prove that $x_{i-1}x_{i+1} \in E(G)$. If $x_{i-1}x_{i+1} \notin E(G)$, then $d(x_{i-1}, x_{i+1}) = 2$. By Lemma 2.3, $N_{G-P_m}(w) \cap (N_{G-P_m}(x_{i-1}) \cup N_{G-P_m}(x_{i+1})) = \emptyset$. Together with (14), we have $|N_G(x_{i-1}) \cup N_G(x_{i+1})| \leq |V(G)| - |N_G[w] - \{x_i\}| \leq n - \delta(G)$, contrary to (1).

We prove $x_{i-1}x_{i+k} \in E(G)$ for $2 \leq k \leq m - i$ by induction. Assume that $x_{i-1}x_{i+k-1} \in E(G)$. If $x_{i-1}x_{i+k} \notin E(G)$, then $d(x_{i-1}, x_{i+k}) = 2$. If $N_{G-P_m}(w) \cap N_{G-P_m}(x_{i+k}) \neq \emptyset$, let $y_1 \in N_{G-P_m}(w) \cap N_{G-P_m}(x_{i+k})$. Then $x_0 \cdots x_{i-1}x_{i+k-1}$ $x_{i+k-2} \cdots x_i w y_1\, x_{i+k}x_{i+k+1} \cdots x_m$ is an (x, y)-path of length $m+2$, contrary to (12). So $N_{G-P_m}(w) \cap N_{G-P_m}(x_{i+k}) = \emptyset$. By Lemma 2.3, $N_{G-P_m}(w) \cap N_{G-P_m}(x_{i-1}) = \emptyset$. Together with (14), we have $|N(x_{i-1}) \cup N(x_{i+k})| \leq |V(G)| - |N_G[w] - \{x_i\}| \leq n - \delta(G)$, contrary to (1).

By symmetry, $x_{i+1}x_{i-k} \in E(G)$ for each k with $0 \leq k \leq i$. □

Let $G_1, \ldots, G_t$ be components of $G[V(G) - V(P_m)]$ and $w \in V(G_1)$. Since $\kappa(G) \geq 2$ and $N_{P_m}(w) = \{x_i\}$, $V(G_1) - \{w\} \neq \emptyset$ and $N_{P_m - x_i}(G_1) \neq \emptyset$. Pick $v \in V(G_1) - \{w\}$ such that

(a) $N_{P_m - x_i}(v) \neq \emptyset$;
(b) subject to (a), $d_{G_1}(w, v)$ is shortest;
(c) subject to (a) and (b), choose $x_k \in N_{P_m - x_i}(v)$ such that $|k - i|$ is as small as possible.

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By symmetry we may assume that $k < i$. Then $k+1 < i+1 \leq m$. Let $ww_1w_2\cdots v$ be a shortest (w, v)-path in G_1. If $d_{G_1}(w, v) = 1$, then $wv \in E(G)$. By Claim 3, $x_{i+1}x_{k+1} \in E(G)$. Then $x_0x_1\cdots x_k vwx_ix_{i-1}\cdots x_{k+1}\ x_{i+1}x_{i+2}\cdots x_m$ is an (x, y)-path of length $m+2$, contrary to (12). So $d_{G_1}(w, v) \geq 2$.

If $d_{G_1}(w, v) \geq 3$, then $d_G(w_2, w) = 2$. We show that $N_{G-P_m}(x_{i+1}) \cap (N_{G-P_m}(w) \cup N_{G-P_m}(w_2)) = \emptyset$. Let $y \in N_{G-P_m}(x_{i+1})$. By Lemma 2.3, $yw \notin E(G)$. If $yw_2 \in E(G)$ and $d(w, v) \geq 4$, then $d_{G_1}(w, y) = 3$, contrary to (b); if $yw_2 \in E(G)$ and $d_{G_1}(w, v) = 3$, then it is contrary to (c) when $k < i-1$, and $x_0x_1\cdots x_k vw_2yx_{i+1}\ x_{i+2}\cdots x_m$, when $k = i-1$, is an (x, y)-path of length $m+2$, contrary to (12). By (14) and (b), we have $N_{P_m}(w) \cup N_{P_m}(w_2) = \{x_i\}$. So $|N_G(w) \cup N_G(w_2)| \leq |V(G)| - |N_G[x_{i+1}] - \{x_i\}| \leq n - \delta(G)$, contrary to (1). Next we assume that $d_{G_1}(w, v) = 2$.

Subcase 3.1 $k < i-1$.

By Claim 3 $x_{k+2}x_{i+1} \in E(G)$. Since $d_{G_1}(w, v) = 2$, then $x_0x_1\cdots x_k vw_1\ wx_ix_{i-1}\cdots x_{k+2}\ x_{i+1}\cdots x_m$ is an (x, y)-path of length $m+2$, contrary to (12).

Subcase 3.2 $k = i-1$.

Fact 1 $N_{P_m}(v) \subseteq \{x_{i-1}, x_i, x_{i+1}\}$.

Suppose by way of contradiction that $\exists x_l \in V(P_m) - \{x_{i-1}, x_i, x_{i+1}\}$ such that $vx_l \in E(G)$. By Claim 3 $x_{l+2}x_{i+1} \in E(G)$ and $x_{i-1}x_{l-2} \in E(G)$. Then $x_0x_1\cdots x_l vw_1wx_ix_{i-1}\ x_{i-2}\cdots x_{l+2}x_{i+1}x_{i+2}\cdots x_m$ when $l \leq i-2$ or $x_0x_1\cdots x_{i-1}\ x_{l-2}x_{l-3}\cdots x_i ww_1vx_lx_{l+1}\cdots x_m$ when $l \geq i+2$ is an (x, y)-path of length $m+2$, contrary to (12).

Fact 2 $x_{i+2} \in V(P_m)$.

Since $m \geq 3$, either $x_{i-2} \in V(P_m)$ or $x_{i+2} \in V(P_m)$. If $x_{i-2} \in V(P_m)$, then $N_{G-P_m}(x_{i-2}) \cap N_{G-P_m}(w) = \emptyset$ by (b) and $N_{G-P_m}(x_{i-2}) \cap N_{G-P_m}(v) = \emptyset$ by Lemma 2.3. So by (14), $N_{G-P_m}(x_{i-2}) \cap (N_{G-P_m}(w) \cup N_{G-P_m}(v)) = \emptyset$. Together with Fact 1, we have $|N(w) \cup N(v)| \leq |V(G)| - |N[x_{i-2}] - \{x_{i-1}, x_i, x_{i+1}\} \cup \{w, v\}| \leq n - \delta(G)$, contrary to (1).

Fact 3 $vx_{i+1} \notin E(G)$.

If $vx_{i+1} \in E(G)$, then $N_{G-P_m}(x_{i+2}) \cap N_{G-P_m}(w) = \emptyset$ by (b) and $N_{G-P_m}(x_{i+2}) \cap N_{G-P_m}(v) = \emptyset$ by Lemma 2.3. By (14), $N_{G-P_m}(x_{i+2}) \cap (N_{G-P_m}(w) \cup N_{G-P_m}(v)) = \emptyset$. Together with Fact 1, we have $|N(w) \cup N(v)| \leq |V(G)| - |N[x_{i+2}] - \{x_{i-1}, x_i, x_{i+1}\} \cup \{w, v\}| \leq n - \delta(G)$, contrary to (1).

Fact 4 There exists $y_1 \in N_{G-P_m}(x_{i+1})$ such that $y_1v \in E(G)$.

By Lemma 2.3, for any $y' \in N_{G-P_m}(x_{i+1})$, $y'w \notin E(G)$. If for any $y' \in N_{G-P_m}(x_{i+1})$, $y'v \notin E(G)$, then together with Facts 1 and 3 we have $|N_G(v) \cup N_G(w)| \leq |V(G)| - |N_G[x_{i+1}] - \{x_{i-1}, x_i\} \cup \{w\}| \leq n - \delta(G)$, contrary to (1). So $\exists y_1 \in N_{G-P_m}(x_{i+1})$ such that $y_1v \in E(G)$.

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Fact 5 $vx_i \notin E(G)$.

If $vx_i \in E(G)$, by Fact 4, $x_0x_1 \cdots x_{i-1}x_i v y_1 x_{i+1}x_{i+2} \cdots x_m$ is an (x, y)-path of length $m + 2$, contrary to (12).

Fact 6 $x_ix_{i+2} \in E(G)$.

If $x_ix_{i+2} \notin E(G)$, then $d(x_i, x_{i+2}) = 2$. Let $y_2 \in N_{G-P_m}(v)$. By Lemma 2.3, $y_2x_i \notin E(G)$. By Claim 3 $x_{i-1}x_{i+1} \in E(G)$ and by Fact 4, if $y_2x_{i+2} \in E(G)$, then $x_0x_1 \cdots x_{i-1}x_{i+1}y_1\, vy_2x_{i+2} \cdots x_m$ is an (x, y)-path of length $m+2$, contrary to (12). Then $N_{G-P_m}(v) \cap (N_{G-P_m}(x_i) \cup N_{G-P_m}(x_{i+2})) = \emptyset$. Together with Facts 1, 3 and 5, we have $|N_G(x_{i+2}) \cup N_G(x_i)| \le |V(G)| - |N_G(v) - \{x_{i-1}\} \cup \{x_i\}| \le n - \delta(G)$, contrary to (1).

By Fact 6, $x_0 \cdots x_{i-1}vw_1wx_ix_{i+2}x_{i+3} \cdots x_m$ is an (x, y)-path of length $m + 2$, contrary to (12). So we excluded both subcases.

Subcase 3.1 and 3.2 can be excluded similarly when $k > i$. □

Proof of Theorem 1.4 By Theorem 3.1, 4.4 and 4.5, either $G \in \{G_1, G_2, G_3, G_4\}$ or G is $[4, n]$-pan-connected. □

Proof of Theorem 1.3 By the structure of G_2 and G_4, for any $x, y \in V(G_4)$, G_2, G_4 both have (x, y)-paths of length 5 and 6. By Theorem 4.5, G_2 and G_4 are both $[5, n]$-pan-connected. Since each graph in $\{G_1, G_3\}$ has a 2-cut, if $\kappa(G) \ge 3$, G is $[5, n]$-pan-connected. □

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