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Manuscript

Ore type condition and geodesic-pancyclic graphs

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MSC: 05C38; 05C45.

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Our terminology is standard concept as indicated. A good reference for any undefined terms is [1]. For a graph G, let $V(G)$ be the vertex set of G and $E(G)$ the edge set, the minimum degree of a graph G is denoted by $\delta(G)$. The complete graph of order n is denoted by K_n. For a subgraph H of a graph G and a subset S of $V(G)$, let $N_H(S)$ be the set of vertices in H that are adjacent to some vertex in S and let the cardinality of set $N_H(S)$ be $|N_H(S)| = d_H(S)$. Furthermore, let $G - H$ and $G[S]$ denote the subgraphs of G induced by $V(G) - V(H)$ and S, respectively. If no ambiguity can arise we sometimes write $N(S)$ instead of $N_G(S)$, δ instead of $\delta(G)$, etc.

A path in a graph G that contains every vertex of G is called a Hamiltonian path of G. A Hamiltonian-connected graph is a graph that each pair of distinct vertices

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u, v in G can be connected by a Hamiltonian path. A $u - v$ path is a path in G that connects u and v. A graph G is said to be $[r, n]$-panconnected if each pair of distinct vertices u, v of G has a $u - v$ path of length k for each k with $r - 1 \leq k \leq n - 1$. In particular, if $r - 1 = d(u, v)$, then graph G is called panconnected.

A shortest path connecting two vertices u and v is called a $u - v$ geodesic. The distance between u and v in a graph G, denoted by $d_G(u, v)$, is the number of edges in a $u - v$ geodesic. A graph G with n vertices is geodesic-pancyclic, if for each pair of vertices $u, v \in V(G)$, every $u - v$ geodesic lies on every cycle of length k satisfying $\max\{2d_G(u, v), 3\} \leq k \leq n$.

Recently, geodesic-pancyclic graphs problems have been studied in some papers [2],[3],[5].

In 2007 Chan et al. proved the following geodesic-pancyclic result by using Dirac type condition.

Theorem 1.1 *(Chan et al. [2]) If G is a connected graph of order $n \geq 4$ and $\delta(G) \geq (n + 2)/2$, then G is geodesic-pancyclic graphs.*

Dirac type condition is a fundamental condition, however, Ore type condition is more powerful fundamental condition.

Thus, in this paper, we study geodesic-pancyclic result by using Ore type condition and prove.

Theorem 1.2 *If G is a connected graph of order $n \geq 4$ and $d(u) + d(v) \geq n + 2$ for each pair of distinct non-adjacent vertices u, v in G, then G is geodesic-pancyclic graphs or $G = \Gamma$.*

Where $\Gamma = K_2^- \vee (K_1 \cup K_m \cup K_{n-m-3}) \cup E^*$, among them K_2^- is a empty subgraph of order 2, $K_2^- \vee (K_1 \cup K_m \cup K_{n-m-3})$ is the graph obtained by taking the join of K_2^- and three disjoint complete graphs K_1, K_m and K_{n-m-3}, E^* is a set of edges whose one of end-vertices are in K_1 and another are in $K_m \cup K_{n-m-3}$, so vertex set $V(\Gamma) = V(K_2^-) \cup V(K_1) \cup V(K_m) \cup V(K_{n-m-3})$ and edge set $E(\Gamma) = E(K_2^- \vee (K_1 \cup K_m \cup K_{n-m-3})) \cup E^*$.

In 2007 Chan et al. also proved the following geodesic-pancyclic graphs result.

Theorem 1.3 *(Chan et al. [2]) If G is a connected graph of order $n \geq 4$ and $d(u) + d(v) \geq (3n - 2)/2$ for each pair of distinct non-adjacent vertices u, v in G, then G is geodesic-pancyclic graphs.*

In this paper, we also prove the following result, which improves the above condition of Theorem 1.3.

Theorem 1.4 *If G is a connected graph of order $n \geq 4$ and $d(u) + d(v) \geq (3n - 3)/2$ for each pair of distinct non-adjacent vertices u, v in G, then G is geodesic-pancyclic graphs or $G \in \{\Gamma, \xi_5, \zeta_5\}$, where ξ_5, ζ_5 are graphs of order 5 that can be found in the proof of Theorem 1.4.*

2 The proof of Main Result

To prove the Theorem 2 and Theorem 4, we need the following lemma.

Lemma 2.1 *(Faudree and Schelp [4]) (1). If G is a connected graph with n vertices and $d(u)+d(v) \geq n+1$ for each pair of distinct non-adjacent vertices u, v in G, then G is $[5, n]$ panconnected. (2). If G is a connected graph with n vertices and $d(u)+d(v) \geq n+2$ for each pair of distinct non-adjacent vertices u, v in G, then G is panconnected.*

Proof of Theorem 1.2.

For any two vertices u, v in G, we consider the following cases.

Case 1. If $uv \notin E(G)$.

In this case, by the condition of Theorem 1.2 that $d(u)+d(v) \geq n+2$, both u and v must have at least a common neighbor in G, i.e., $d_G(u, v) \leq 2$, this implies $d_G(u, v) = 2$.

Then let uwv be a path of order 3 and let $H = G - w$, by $d(u)+d(v) \geq n+2$, we can check that $d_H(x)+d_H(y) \geq |V(H)|+1$ for each pair of distinct non-adjacent vertices x, y in H, and by Lemma 1, there exist $u-v$ paths in H of each lengths k with $4 \leq k \leq |V(H)|-1 = n-2$. Since $d_H(x)+d_H(y) \geq |V(H)|+1$, so $|N_H(x) \cap N_H(y)| \geq 1$, i.e. there also exists a $u-v$ path of length 2, so to complete the proof, we only need to prove there exists a $u-v$ path of length 3.

If there does not exist any $u-v$ paths of length 3 in H, then every vertex of $N_H(u)$ does not adjacent to any vertex of $N_H(v)$. Since $d_H(x)+d_H(y) \geq |V(H)|+1$ for all $x \in N_H(u), y \in N_H(v)$, this implies that $G[N_H(u)]$ and $G[N_H(v)]$ are two complete subgraphs, and by $d_H(x)+d_H(y) \geq |V(H)|+1$, x and y all must be adjacent to u and v for $x \in N_H(u), y \in N_H(v)$. Clearly, in this case, w may be adjacent to some vertices of $N_H(u) \cup N_H(v)$, we denote the graph by $K_2^- \vee (K_1 \cup K_m \cup K_{n-m-3}) \cup E^* = \Gamma$, where $K_2^- = G[\{u, v\}]$ is a empty subgraph, $K_1 = w, K_m = G[N_H(u)]$ and $K_{n-m-3} = G[N_H(v)]$, E^* is an edge set that connects from vertex w to some vertices of $K_m \cup K_{n-m-3}$.

Case 2. If $uv \in E(G)$.

In this case by condition of Theorem 1.2 that $d(u)+d(v) \geq n+2$ and by Lemma 2.1 such that G is panconnected, so the edge uv lies on every cycle of length k satisfying $3 \leq k \leq n$.

Therefore, this complete the proof of Theorem 1.2.

Proof of Theorem 1.4.

For any two vertices u, v in G, we consider the following cases on order n.

When $n \geq 7$, by the condition of Theorem 1.4 that $d(u)+d(v) \geq (3n-3)/2$ for each pair u, v in $V(G)$ of $uv \notin E(G)$, we can check that $d(u)+d(v) \geq (3n-3)/2 \geq n+2$, by Theorem 1.2, G is geodesic-pancyclic graphs or $G = \Gamma$.

When $n = 6$, by the condition of Theorem 1.4 that $d(u)+d(v) \geq (3n-3)/2$ for each pair of nonadjacent vertices u, v in $V(G)$, we have $d(u)+d(v) \geq (3n-3)/2 = 15/2$.

Since $d(u)+d(v)$ is integral, so $d(u)+d(v)8 = n+2$, by Theorem 2, G is panconnected, so G is geodesic-pancyclic graphs or $G = \Gamma$.

When $n = 5, d(u) + d(v) \geq (3n - 3)/2 = 6$.

In this case, since $d(u) + d(v) \geq (3n - 3)/2$, so $d(u, v) \leq 2$.

If $d(u, v) = 2$, let $\{x, y, w\} = V(G - uwv)$, by $d(u) + d(v) \geq 6$, w, x and y all are adjacent to u and v. For any uwy path of order 3, when there exists a $u - v$ path of order 4 in $G - w$, then the proof is complete. When there does not exist any $u - v$ path of order 4 in $G - w$, then $xy \notin E(G)$, in this case G is the graph satisfying $N(x) = N(y) = \{u, v, w\}$, $N(w) = \{u, v, x, y\}$ and $N(u) = N(v) = \{w, x, y\}$, denoted by ξ_5.

If $d(u, v) = 1$. In this case, since $d(u) + d(v) \geq 6 = n + 1$, by Lemma 2.1, we have a $u - v$ path of order 5, denoted by $uxwyv$. Clearly, we can prove that uw or $vw \in E(G)$ (Otherwise, if $uw, vw \notin E(G)$, then $d(w) = 2$, so we have $d(u)+d(w) < 6$, a contradiction), thus, there exists a $u-v$ path of order 4. Next, if uy or $vx \in E(G)$, then we have $u - v$ path of order 3. If $uy, vx \notin E(G)$, by $d(u) + d(y) \geq 6$ and $d(x) + d(v) \geq 6$, we must see that $uw, vw, xy \in E(G)$, in this case G is the graph satisfying $N(u) = N(y) = \{u, w, y\}$, $N(x) = N(v) = \{u, w, y\}$, $N(w) = \{u, v, x, y\}$, denoted by ζ_5.

When $n = 4$, by the condition of Theorem that $d(u) + d(v) \geq (3n - 3)/2$ for each pair of nonadjacent vertices u, v in $V(G)$, we have $d(u) + d(v) \geq (3n - 3)/2 = 9/2$. Since $d(u) + d(v)$ is integral, so $d(u) + d(v) \geq 5$. For order $n = 4$, so $d(u) \geq 3$ for any $u \in V(G)$. In this case if $xy \notin E(G)$, then $d(x) + d(y) \leq 4$. This implies G is a complete graph K_4, so G is geodesic-pancyclic graph.

Therefore, this complete the proof of Theorem 1.4.

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