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Reflection positivity, rank connectivity, and homomorphism of graphs. (English)

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This interesting paper shows that a graph parameter can be realized as the number of homomorphisms into a fixed (weighted) graph if and only if it satisfies two linear algebraic conditions: reflection positivity and exponential rank connectivity. In terms of statistical physics, this can be viewed as a characterization of partition functions of vertex coloring models.

That is, if G is an unweighed graph and H is a weighted graph with positive real weight $\alpha_H(i)$ associated with each node i and a real weight $\beta_H(i,j)$ associated with each edge ij. To every homomorphism $\phi : V(G) \to V(H)$, we assign the weights

$$\alpha_\phi = \prod_{u \in V(G)} \alpha_H(\phi(u))$$

and

$$\hom_\phi(G,H) = \prod_{uv \in E(G)} \beta_H(\phi(u),\phi(v)),$$

and define

$$\hom(G,H) = \sum_{\phi: V(G) \to V(H)} \alpha_\phi \hom_\phi(G,H).$$

The definition shows a graph parameter denoted by $\hom(.,H)$.

In this paper, the authors prove that a graph parameter f can be represented in form $f = \hom(.,H)$ for some finite weighted graph H on at most q nodes if and only if it is reflection positive and its connection matrices rank $r(f,k) \leq q^k$ for all $k \geq 0$.

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Keywords : graph homomorphism; partition function; connection matrix; reflection positivity; rank connectivity; homomorphism of graphs; graph parameters; vertex coloring models

Classification :

∗05C99 Graph theory
82B99 Equilibrium statistical mechanics
05C15 Chromatic theory of graphs and maps