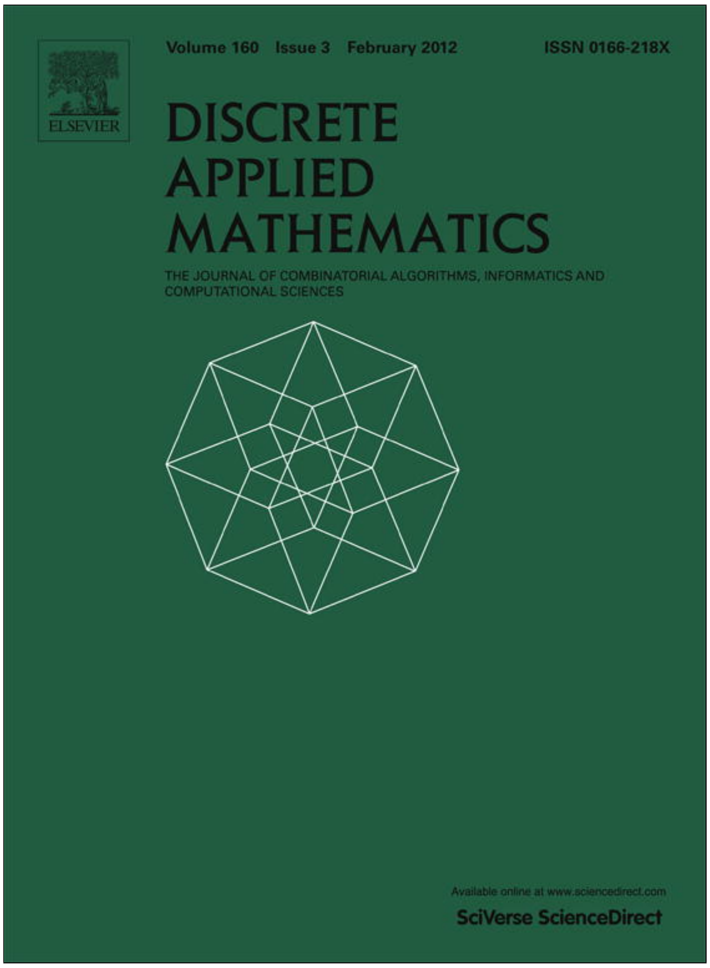
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Pancyclic graphs and degree sum and neighborhood union involving distance two

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ABSTRACT

For a graph G, δ denotes the minimum degree of G. In 1971, Bondy proved that, if G is a 2-connected graph of order n and $d(x) + d(y) \geq n$ for each pair of non-adjacent vertices x, y in G, then G is pancyclic or $G = K_{n/2,n/2}$. In 2006, Wu et al. proved that, if G is a 2-connected graph of order $n \geq 6$ and $|N(x) \cup N(y)| + \delta \geq n$ for each pair of non-adjacent vertices x, y of $d(x, y) = 2$ in G, then G is pancyclic or $G = K_{n/2,n/2}$. In this paper, we introduce a new condition which generalizes two conditions of degree sum and neighborhood union and prove that, if G is a 2-connected graph of order $n \geq 6$ and $|N(x) \cup N(y)| + d(w) \geq n$ for any three vertices x, y, w of $d(x, y) = 2$ and wx or $wy \notin E(G)$ in G, then G is pancyclic or $G = K_{n/2,n/2}$. This result also generalizes the above two results.

1. Introduction

We generalize two well-known degree sum and neighborhood union conditions for the characterizing of Hamiltonian graphs, in particular for pancyclic graphs. First, we give a few definitions and some notation. We consider only finite undirected graphs with no loops or multiples. We denote by $\delta(G)$ the minimum degree of G. If u is a vertex and H is a subgraph of G, then define $N_H(u) = \{v \in V(H) : uv \in E(G)\}$ to be the vertex set of H that is adjacent to vertex u, and set $N_H[u] = N_H(u) \cup \{u\}$. Let $G - H$ and $G[S]$ denote the subgraphs of G induced by $V(G) - V(H)$ and S, respectively. If $C_m = x_1x_2 \cdots x_mx_1$ is a cycle of order m, let $N^+_{C_m}(u) = \{x_{i+1} : x_i \in N_{C_m}(u)\}$, $N^-_{C_m}(u) = \{x_{i-1} : x_i \in N_{C_m}(u)\}$, and $N^{\pm}c_m(u) = N^+_{C_m}(u) \cup N^-_{C_m}(u)$, where subscripts are taken modulo m. For a graph G of order n, in 1960, Ore introduced the degree sum condition $d(u) + d(v) \geq n$ for G to be Hamiltonian; in 1987, Faudree et al. introduced the neighborhood union $NC = \min\{|N(x) \cup N(y)| : x, y \in V(G), xy \notin E(G)\}$; and, in 1989, Lindquester [12] introduced the neighborhood union of each pair of vertices at distance 2 as follows: $NC_2 = \min\{|N(x) \cup N(y)| : x, y \in V(G), d(x, y) = 2\}$. In this paper, we introduce a new sufficient condition of generalizing degree sum and neighborhood union as follows: $DNC_2 = \min\{|N(x) \cup N(y)| + d(w) : x, y, w \in V(G), d(x, y) = 2, wx \text{ or } wy \notin E(G)\}$. For graphs A and B, the join operator $A \vee B$ of A and B is the graph constructed from A and B by adding all edges joining the vertices of A and the vertices of B. If no ambiguity can arise, we sometimes write $N(u)$ instead of $N_G(u)$, δ instead of $\delta(G)$, etc.

If a graph G has a *Hamiltonian* cycle (a cycle containing all vertices of G), then G is said to be Hamiltonian. A graph G is said to be *pancyclic* if G contains cycles of every length k, $3 \leq k \leq n$. Other terminology and notation not defined here can be found in Gould [3].

In 1960, Ore obtained the following well-known Hamiltonian result on degree sum.

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Theorem 1.1 (*Ore [13]*). *If G is a 2-connected graph of order n and $d(x) + d(y) \geq n$ for each pair of non-adjacent vertices x, y in G, then G is Hamiltonian.*

In 1971, Bondy considered Ore's condition for pancyclic graphs.

Theorem 1.2 (*Bondy [1]*). *If G is a 2-connected graph of order n and $d(x) + d(y) \geq n$ for each pair of non-adjacent vertices x, y in G, then G is pancyclic or $G = K_{n/2,n/2}$.*

In 1991, Faudree et al. proved the following result on neighborhood union.

Theorem 1.3 (*Faudree et al. [2]*). *If G is a 2-connected graph of order n and $|N(x) \cup N(y)| + \delta \geq n$ for each pair of non-adjacent vertices x, y in G, then G is Hamiltonian.*

In 2006, Wu et al. proved the following pancyclic result on neighborhood union at distance 2.

Theorem 1.4 (*Wu et al. [14]*). *If G is a 2-connected graph of order $n \geq 6$ and $|N(x) \cup N(y)| + \delta \geq n$ for each pair of non-adjacent vertices x, y of $d(x, y) = 2$ in G, then G is pancyclic or $G = K_{n/2,n/2}$.*

In this paper, we present a new sufficient condition which generalizes the two well-known degree sum and neighborhood union conditions and prove the following result.

Theorem 1.5. *If G is a 2-connected graph of order $n \geq 6$ and $|N(x) \cup N(y)| + d(w) \geq n$ for any three vertices x, y, w of $d(x, y) = 2$ and wx or $wy \notin E(G)$ in G, then G is pancyclic or $G = K_{n/2,n/2}$.*

Note. If each pair of nonadjacent vertices of a graph G satisfies the condition of Theorem 1.2, then, clearly, for any three vertices x, y, w of $d(x, y) = 2$ and $xw \notin E(G)$ or $yw \notin E(G)$, $|N(x) \cup N(y)| + d(w) \geq n$ holds, by Theorem 1.5, G is pancyclic or $G = K_{n/2,n/2}$; thus Theorem 1.5 implies Theorem 1.2. Also, if each pair of nonadjacent vertices of a graph G satisfies the condition of Theorem 1.4, then, clearly, for any three vertices x, y, w of $d(x, y) = 2$ and $xw \notin E(G)$ or $yw \notin E(G)$, $|N(x) \cup N(y)| + d(w) \geq n$ holds, by Theorem 1.5, G is pancyclic or $G = K_{n/2,n/2}$; thus, Theorem 1.5 implies Theorem 1.4.

Corollary 1.6. *If G is a 2-connected graph of order $n \geq 3$ and $|N(x) \cup N(y)| + d(w) \geq n$ for any three vertices x, y, w of $d(x, y) = 2$ and wx or $wy \notin E(G)$ in G, then G is Hamiltonian.*

2. Proof of the main result

Obviously, Theorem 1.5 can be obtained immediately by the following Lemmas 2.1 and 2.6.

Lemma 2.1. *If G is a 2-connected graph of order $n \geq 6$ and $DNC_2 \geq n$, then G has C_3, C_4 or $G = K_{n/2,n/2}$.*

Proof. We consider the following two cases.

Case 1. There exists at least a vertex u of G satisfying that the degree number of u is more than 2.

Subcase 1.1. $N(u)$ has two adjacent vertices v, w.

In this case, clearly, G has C_3. Then we will prove that G has C_4. Otherwise, if G does not have C_4, then, clearly, $G[N(u)]$ does not have a path of order 3. Let $z \in N(u) \setminus \{v, w\}$; since G is 2-connected, w or v must be adjacent to some vertex of $G - N[u]$. Without loss of generality, assume that w is adjacent to some $x \in V(G - N[u])$; then, since G does not have C_4, we have the following: both v and x do not have a common neighbor in $G - N[u]$; any two distinct vertices of $N(u)$ do not have a common neighbor in $G - N[u]$; $zx \notin E(G)$. We consider the following two cases of the distance of vertices z and x. (1) When $d(z, x) = 2$, clearly, $N(v) \cap N(\{x, z\}) = \{u, w\}$, and each of $\{x, z, v\}$ is not adjacent to any of $\{x, z, v\}$, so we can check that $|N(x) \cup N(z)| + d(v) \leq |V(G)| - |\{x, z, v\}| + |\{u, w\}| \leq n - 1$, which contradicts the assumption of Lemma 2.1. (2). When $d(z, x) \neq 2$, similarly, we can check that $|N(v) \cup N(z)| + d(x) \leq |V(G)| - |\{x, z, v\}| + |\{w\}| \leq n - 1$, a contradiction.

Subcase 1.2. $N(u)$ does not have two adjacent vertices.

Let $v, w, z \in N(u)$. Since $d(w, z) = 2$, by the condition of the lemma that $|N(w) \cup N(z)| + d(v) \geq n$, we can check that $|V(G - N[u])| \geq |N(u)| - 1$.

(1) $|V(G - N[u])| = |N(u)| - 1$.

In this case, using $|N(w) \cup N(z)| + d(v) \geq n$ for any three vertices w, z, v in $N(u)$, each vertex of $N(u)$ must be adjacent to every vertex of $G - N(u)$ (for example, if $v \in N(u)$ is not adjacent to some vertex of $G - N(u)$, let $w, z \in N(u) \setminus \{v\}$; then we can check that $|N(w) \cup N(z)| + d(v) \leq n - 1$, a contradiction), so $G \in N(u) \vee (G - N(u))$, where "$\vee$" is the join operator. Then, if G does not have C_3, then $G - N(u)$ is an empty subgraph; this implies that $G = K_{n/2,n/2}$. If G has C_3, then, clearly, G contains C_3, C_4.

(2) $|V(G - N[u])| \geq |N(u)|$.

If G does not have C_3, then, for any two vertices x, y of distance 2 in $V(G - N[u])$, using $|N(x) \cup N(y)| + d(u) \geq n$, we can check that there are at most $|N(u)| - 1$ vertices of $G - N[u]$ that are not adjacent to x and y, so there are at least $|V(G - N[u])| - (|N(u)| - 1)$ vertices of $G - N[u]$ that are adjacent to x or y.

Clearly, each vertex v of $N(u)$ must be adjacent to at least two vertices of $G-N[u]$. Otherwise, if v is at most adjacent to a vertex of $G-N[u]$, letting $w, z \in N(u) \setminus \{v\}$, we can check that $|N(w) \cup N(z)| + d(v) \le n-1$, a contradiction. Since G does not have C_3, if v of $N(u)$ is adjacent to two vertices x, y of $G-N[u]$, then $xy \notin E(G)$, and vertex v is not adjacent to any of $N(x) \cup N(y)$.

Since G is 2-connected, $G-N[u]$ is not empty subgraph. Let q, r be two adjacent vertices of $G-N[u]$; if there exist two vertices w, z of $N(u)$ satisfying that both w, z are not adjacent to q (or r), then we have $|N(w) \cup N(z)| \le (|V(G-N[u])|-1) + |\{u\}|$. Letting $v \in N(u) \setminus \{w, z\}$ be adjacent to two vertices x, y of $G-N[u]$, we can check that $d(v) \le |V(G-N[u])| - |N_{G-N[u]}(x) \cup N_{G-N[u]}(y)| + |\{u\}| \le |V(G-N[u])| - (|V(G-N[u])| - (|N(u)|-1)) + |\{u\}| \le |N(u)|$. So we have $|N(w) \cup N(z)| + d(v) \le (|V(G-N[u])|-1) + |\{u\}| + |N(u)| \le |V(G-N(u) \cup \{u\})| + |N(u)| \le n-1$, a contradiction. This contradiction shows that q, like r, must be adjacent to at least one of $\{w, z\}$. Since $|N(u)| \ge 3$, there exists at least a vertex of $N(u)$ that is adjacent to q and r, so we have C_3, a contradiction. This contradiction shows that G contains C_3. It is also easy to see that there must exist two vertices of $N(u)$ that have a common neighbor in $G-N[u]$. Otherwise, we have $|N(w) \cup N(z)| + d(v) \le |V(G-N[u])| + 2|\{u\}| \le n-1$ for any three vertices w, z, v in $N(u)$, a contradiction. Without loss of generality, assume that $w, z \in N(u)$ are adjacent to vertex x of $G-N[u]$, so G also contains $C_4 = uwxzu$, i.e., G contains C_3, C_4.

Case 2. $d(u) = 2$ for each $u \in V(G)$.

In this case, G is the cycle $C_n = x_1x_2\cdots x_nx_1$ of order n. Since $n \ge 6$, we have $|N(x_1) \cup N(x_3)| + d(x_5) \le n-1$, a contradiction. □

Lemma 2.2. *If G is a 2-connected graph of order $n \ge 6$, and $DNC_2 \ge n$, C_m is a cycle of order m, u is a vertex of $G-C_m$, and $|N_{C_m}(u)| \ge 2$, then the two following conditions hold.*

(1) If $x_{i+1}, x_{j+1} \in N^+_{C_m}(u)$ and $d(x_{i+1}, x_{j+1}) \le 2$ and x_{i+1}, x_{j+1} are not adjacent to any of $N[u] \setminus V(C_m)$, then there exists $x_k \in N_{C_m}(u)$ satisfying that $x_{i+1}x_{k+1}$ or $x_{j+1}x_{k+1} \in E(G)$.

(2) If there exist $x_{i+1}, x_{j+1} \in N^+_{C_m}(u)$ satisfying that $d(x_{i+1}, x_{j+1}) \ge 3$ and $\{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\} \cap N_{C_m}(u) = \emptyset$, and x_{j+1} is not adjacent to any of $N[u] \setminus V(C_m)$, then there exists at least a vertex x_k in $P = x_{j+1}x_{j+2}\cdots x_i$ such that $x_k \in N(x_{j+1})$ with $x_{k-1}x_{i+1}$ or $x_{k-1}u \in E(G)$.

Clearly, (1) and (2) together imply that $V(C_m) \cup \{u\}$ structures a C_{m+1}.

Proof. First, we consider (1), i.e., that $d(x_{i+1}, x_{j+1}) \le 2$. (i) When $d(x_{i+1}, x_{j+1}) = 1$, we can take x_{k+1} as x_{i+1} or x_{j+1}, so in this case (1) of Lemma 2.2 holds. (ii) When $d(x_{i+1}, x_{j+1}) = 2$, suppose (1) of lemma is false, i.e., for any $x \in N(u) \cup \{u\}$, when $x \notin V(C_m)$, x is not adjacent to x_{i+1}, x_{j+1}. When $x = x_k \in V(C_m)$, x_{k+1} is not adjacent to x_{i+1}, x_{j+1}, this implies that $|N(x_{i+1}) \cup N(x_{j+1})| \le n - |N(u) \cup \{u\}|$, so $|N(x_{i+1}) \cup N(x_{j+1})| + d(u) \le n-1$, a contradiction.

Therefore, (1) holds, and we can construct cycle $C_{m+1} = x_iux_kx_{k-1}\cdots x_{i+1}x_{k+1}x_{k+2}\cdots x_i$ if $x_{k+1}x_{i+1} \in E(G)$ for some $x_k \in N_{C_m}(u)$ or cycle $C_{m+1} = x_jux_kx_{k-1}\cdots x_{j+1}x_{k+1}x_{k+2}\cdots x_j$ if $x_{k+1}x_{j+1} \in E(G)$ for some $x_k \in N_{C_m}(u)$, two cycles both consisting of u and C_m.

If (2) is false, i.e., for any $x \in N(x_{j+1})$, when $x \notin V(C_m)$, x is not adjacent to vertex u, and, since $d(x_{i+1}, x_{j+1}) \ge 3$, x is also not adjacent to x_{i+1}. When $x = x_k$ in path $P = x_{j+1}x_{j+2}\cdots x_i$, then $x_{k-1}x_{i+1}, x_{k-1}u \notin E(G)$, i.e., none of $N^-_P(x_{j+1})$ is adjacent to x_{i+1}, u. When x in path $R = x_{i+1}x_{i+2}\cdots x_{j-1}$, since $d(x_{i+1}, x_{j+1}) \ge 3$ and $\{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\} \cap N_{C_m}(u) = \emptyset$, x is not adjacent to u, x_{i+1}, i.e., none of $N_R(x_{j+1})$ is adjacent to x_{i+1}, u. Also x_{i+1}, u are not adjacent to x_{i+1}, u. Clearly in the three sets $N^-_P(x_{j+1})$, $N_R(x_{j+1})$ and $\{u, x_{i+1}\}$, there are no two sets that have any common vertex, and $|N_R(x_{j+1})| + |N^-_P(x_{j+1})| = |N_{C_m}(x_{j+1})| - 1$. Hence we can check that $|N(u) \cup N(x_{i+1})| \le n - |N_R(x_{j+1})| - |N^-_P(x_{j+1})| - |\{x_{i+1}, u\}| \le n - d(x_{j+1}) - 1$; this implies that $|N(u) \cup N(x_{i+1})| + d(x_{j+1}) \le n-1$, a contradiction.

Therefore, (2) holds, and we can construct cycle $C_{m+1} =: x_iux_jx_{j-1}\cdots x_{i+1}x_{k-1}x_{k-2}\cdots x_{j+1}x_kx_{k+1}\cdots x_i$ or $x_jux_{k-1}x_{k-2}\cdots x_{j+1}x_kx_{k+1}\cdots x_j$, respectively.

Similarly, it is also easy to obtain the following result by considering the reverse direction on C_m from Lemma 2.2. □

Corollary 2.3. *If G is a 2-connected graph of order $n \ge 6$, and $DNC_2 \ge n$, C_m is a cycle of order m, u is a vertex of $G-C_m$, and $|N_{C_m}(u)| \ge 2$, then the two following conditions hold.*

(1) *If $x_{i-1}, x_{j-1} \in N^-_{C_m}(u)$ and $d(x_{i-1}, x_{j-1}) \le 2$ and x_{i-1}, x_{j-1} are not adjacent to any of $N[u] \setminus V(C_m)$, then there exists $x_k \in N_{C_m}(u)$ satisfying that $x_{i-1}x_{k-1}$ or $x_{j-1}x_{k-1} \in E(G)$.*

(2) *If there exist $x_{i-1}, x_{j-1} \in N^-_{C_m}(u)$ satisfying that $d(x_{i-1}, x_{j-1}) \ge 3$ and $\{x_{j+1}, x_{j+2}, \ldots, x_{i-1}\} \cap N_{C_m}(u) = \emptyset$ and x_{j-1} is not adjacent to any of $N[u] \setminus V(C_m)$, then there exists at least a vertex x_k in $P = x_{j+1}x_{j+2}\cdots x_i$ such that $x_k \in N(x_{j+1})$ with $x_{k+1}x_{i+1}$ or $x_{k+1}u \in E(G)$.*

Now we prove the following Lemma 2.6. First, we state two propositions we need.

Proposition 2.4. *Let $C_{m+1} = y_1y_2\cdots y_{m+1}y_1$ be the cycle of order $m+1$ obtained from (1) or (2) of Lemma 2.2. If $v \in V(G-C_{m+1})$ is adjacent to some y_h in $V(C_{m+1}) \cap V(C_m)$, when $y_h \in \{x_i, x_{i+1}, x_j, x_{j+1}, x_{k-1}, x_k, x_{k+1}\}$ described in Lemma 2.2, then, clearly, y_{h+1} or $y_{h-1} \in N^{\pm}_{C_m}(v)$. When $y_h \notin \{x_i, x_{i+1}, x_j, x_{j+1}, x_k, x_{k+1}\}$, then, clearly, $y_{h+1}, y_{h-1} \in N^{\pm}_{C_m}(v)$.*

Proposition 2.5. *Let G be a 2-connected graph of order $n \geq 6$ and $DNC_2 \geq n$, and let C_m be a cycle of order m. If $C_{m+1} = y_1y_2 \cdots y_{m+1}y_1$ is the cycle of order $m+1$ obtained from (1) or (2) of Lemma 2.2 consisting of u and C_m, and if each $w \in N^{\pm}_{C_m}(y)$ is not adjacent to any of $N(y) \setminus V(C_m)$, where $u, y \in V(G - C_m)$, and if there exists $x_j \in N_{C_m}(y)$ satisfying that this path $x_{j-1}x_jx_{j+1}$ of C_m is also a path of C_{m+1}, and if there exists $y_h \in N_{C_{m+1}}(y) \setminus \{x_j\}$ satisfying that y_{h+1} or $y_{h-1} \in N^{\pm}_{C_m}(y)$, then we can obtain that C_{m+2} consists of y and C_{m+1}.*

That is, under the hypothesis of Proposition 2.5, there must exist $y_k, y_h \in N_{C_{m+1}}(y)$ satisfying that both y_{k+1}, y_{h+1} or both y_{k-1}, y_{h-1} are not adjacent to any of $N_{G-C_{m+1}}(y)$. (For example, let $x_{j-1}x_jx_{j+1} = y_{i-1}y_iy_{i+1}$. If $y_{h+1} \in N^{\pm}_{C_m}(y)$, then y_{i+1}, y_{h+1} are not adjacent to any of $N(y) \setminus V(C_{m+1})$; if $y_{h-1} \in N^{\pm}_{C_m}(y)$, then y_{i-1}, y_{h-1} are not adjacent to any of $N(y) \setminus V(C_{m+1})$.) Thus, the cycle C_{m+1} and graph G must satisfy the condition of Lemma 2.2, so we can obtain C_{m+2} consisting of y and C_{m+1} immediately by Lemma 2.2.

Lemma 2.6. *If G is a 2-connected graph of order $n \geq 6$ and $DNC_2 \geq n$, and G has C_m, C_{m+1}, where $m \leq n-2$, then G has C_{m+2}.*

Proof. Assume, to the contrary, that G does not have C_{m+2}.

Under the hypothesis, for each vertex x of $G - C_m$, then, clearly, none of $N^{\pm}_{C_m}(x)$ are adjacent to $N(x) \setminus V(C_m)$ (for example, if there exists vertex x_i of $C_m = x_1x_2 \cdots x_mx_1$ adjacent to x, and x_{i+1} or x_{i-1} is adjacent to $y \in N(x) \setminus V(C_m)$, then we obtain cycle $x_ixyx_{i+1}x_{i+2} \cdots x_i$ or cycle $x_ixyx_{i-1}x_{i-2} \cdots x_i$; all are C_{m+2} consisting of $V(C_m) \cup \{x, y\}$, a contradiction). Then we consider the following cases.

Case 1. There exist x, y in $G - C_m$ such that $|N_{C_m}(x)| \geq 2$ and $|N_{C_m}(y)| \geq 2$.

Subcase 1.1. $N_{C_m}(x) = N_{C_m}(y) = \{x_i, x_j\}$.

In this case, we have $xy \in E(G)$. Otherwise, if $xy \notin E(G)$, let $x_k \in V(C_m) \setminus \{x_i, x_j\}$ satisfying that $x_k \in \{x_{i+1}, x_{j+1}\}$; then, we can check that $|N(x) \cup N(y)| \leq n - |N[x_k] \setminus \{x_i, x_j\}| - |\{x, y\}| \leq n - d(x_k) - 1$, which implies that $|N(x) \cup N(y)| + d(x_k) \leq n-1$, a contradiction.

Thus, $xy \in E(G)$; then, by (1) or (2) of Lemma 2.2, we construct C_{m+1} by $V(C_m) \cup \{x\}$. Clearly, C_{m+1} contains xx_i or xx_j. Since y is adjacent to x and x_i, x_j, we have C_{m+2}.

Subcase 1.2. $N_{C_m}(x) \neq N_{C_m}(y)$ or $\max\{|N_{C_m}(x)|, |N_{C_m}(y)|\} \geq 3$.

Subcase 1.2.1. $xy \in E(G)$.

Subcase 1.2.1.1. $N_{C_m}(x) \cap N_{C_m}(y) \neq \emptyset$.

In this case, let $x_j \in N_{C_m}(x) \cap N_{C_m}(y)$. We choose $x_i \in N_{C_m}(x)$ satisfying that $\{x_{i+1}, x_{i+2}, \cdots, x_{j-1}\} \cap N_{C_m}(x) = \emptyset$. (I) If $d(x_{i+1}, x_{j+1}) \geq 3$, by (2) of Lemma 2.2, then there exists $x_h \in N_{C_m}(x_{j+1})$ satisfying that $x_{i+1}x_{h-1}$ or $xx_{h-1} \in E(G)$. When $x_{i+1}x_{h-1}$, then $C_{m+2} = x_ixyx_jx_{j-1} \cdots x_{i+1}x_{h-1}x_{h-2} \cdots x_{j+1}x_hx_{h+1} \cdots x_i$. When xx_{h-1}, then $C_{m+2} = x_jyxx_{h-1}x_{h-2} \cdots x_{j+1}x_hx_{h+1} \cdots x_j$. (II) If $d(x_{i+1}, x_{j+1}) = 2$, by (1) of Lemma 2.2, then there exists $x_h \in N_{C_m}(x)$ satisfying that $x_{j+1}x_{h+1}$ or $x_{i+1}x_{h+1} \in E(G)$. When $x_{j+1}x_{h+1} \in E(G)$, then $C_{m+2} = x_jyxx_hx_{h-1} \cdots x_{j+1}x_{h+1}x_{h+2} \cdots x_j$. When $x_{i+1}x_{h+1} \in E(G)$, we construct $C_{m+1} = y_1y_2 \cdots y_{m+1}y_1$ by (1) of Lemma 2.2, Clearly, in this case, path $x_{j-1}x_jx_{j+1}$ of C_m is also a path of C_{m+1}, together with Proposition 2.5 and since $|N_{C_m}(y)| \geq 2$, so there must exist vertex $y_r \in N_{C_{m+1}}(y) \setminus \{x_j\}$ and without loss of generality, assume that x_j of C_m is labeled as some y_h on C_{m+1} satisfying $\{y_{h+1}, y_{h+2}, \ldots, y_{r-1}\} \cap N_{C_{m+1}}(y) = \emptyset$, and both y_{r+1}, y_{h+1} or both y_{r-1}, y_{h-1} are not adjacent to any of $N(y) \setminus V(C_{m+1})$, by (1) or (2) of Lemma 2.2, we can obtain C_{m+2}.

Subcase 1.2.1.2. $N_{C_m}(x) \cap N_{C_m}(y) = \emptyset$.

Subcase 1.2.1.2.1. There exist consecutive x_i, x_{i+1} on C_m satisfying that $x_i, x_{i+1} \in N_{C_m}(x)$ or $N_{C_m}(y)$.

Without loss of generality, assume that $x_i, x_{i+1} \in N_{C_m}(x)$, so we construct $C_{m+1} = x_ixx_{i+1}x_{i+2} \cdots x_i$. Then, for each $x_j \in N_{C_m}(y)$, path $x_{j-1}x_jx_{j+1}$ of C_m is also a path of C_{m+1}. Since $|N_{C_m}(y)| \geq 2$, by Proposition 2.5, we can obtain C_{m+2}.

Subcase 1.2.1.2.2. Subcase 1.2.1.2.1 does not exist

Using x and C_m we first construct C_{m+1}.

By hypothesis of not C_{m+2} and not Subcase 1.2.1.2.1, together with xy is edge, so for any $x_i \in N_{C_m}(x)$, y is not adjacent to x_{i-1}, x_i, x_{i+1}. Since $|N_{C_m}(y)| \geq 2$, at least there exists a vertex $x_j \in N_{C_m}(y)$ satisfying that path $x_{j-1}x_jx_{j+1}$ of C_m is also a path of C_{m+1}, by (1) or (2) of Lemma 2.2, we can obtain C_{m+2}.

Subcase 1.2.2. $xy \notin E(G)$.

Subcase 1.2.2.1. There exist consecutive x_i, x_{i+1} on C_m satisfying that $x_i, x_{i+1} \in N_{C_m}(x)$ or $N_{C_m}(y)$.

Without loss of generality, assume that $x_i, x_{i+1} \in N_{C_m}(x)$. (i) If $N_{C_m}(y) \neq \{x_i, x_{i+1}\}$, we construct $C_{m+1} = x_ixx_{i+1}x_{i+2} \cdots x_i$. Then if y is adjacent to two consecutive vertices of C_{m+1}, we have C_{m+2}. Otherwise, if y is not adjacent to any two consecutive vertices of C_{m+1}. Since this is not Subcase 1.1, there exists at least $x_j \in N_{C_m}(y)$ satisfying that path $x_{j-1}x_jx_{j+1}$ of C_m is also a path of C_{m+1}, by Proposition 2.5, we have C_{m+2}. (ii) If $N_{C_m}(y) = x_i, x_{i+1}$, we construct $C_{m+1} = x_iyx_{i+1}x_{i+2} \cdots x_i$ consisting of C_m and y. Then, if x is adjacent to two consecutive vertices of C_{m+1}, we have C_{m+2}. Otherwise, since this is not Subcase 1.1, there exists at least $x_j \in N_{C_m}(x)$ satisfying that path $x_{j-1}x_jx_{j+1}$ of C_m is also a path of C_{m+1}, by Proposition 2.5, we have C_{m+2} consisting of C_{m+1} and x.

Subcase 1.2.2.2. Subcase 1.2.2.1 does not exist.

In this case, if $d(x_{h+1}, x_{k+1}) \geq 3$ for some two vertices $x_{h+1}, x_{k+1} \in N^{+}_{C_m}(x)$, then there must exist at least three vertices in $N^{+}_{C_m}(x)$ that are adjacent to x_{k+1} (otherwise, since x, x_{h+1}, x_{k+1} is a independent vertex set and since $d(x_{h+1}, x_{k+1}) \geq 3$, x_{h+1}

and any of x, x_{h+1} do not have any common neighbor in $G - C_m$, so we can check that $|N(x) \cup N(x_{h+1})| + d(x_{k+1}) \le n - 1$, a contradiction). We consider the following cases.

(1) When $x_i \in N_{C_m}(x) \cap N_{C_m}(y)$. Since this is not Subcase 1.2.2.1, $xx_{i+1}, yx_{i+1} \notin E(G)$, and since each of $\{x, y\}$ and x_{i+1} do not have any common neighbor in $G-C_m$ (otherwise, it is easy to obtain a C_{m+2}, a contradiction). By the assumption of the lemma that $|N(x)\cup N(y)|+d(x_{i+1}) \ge n$ and $\{x, y, x_{i+1}\}$ is a independent vertex set, $|N(x)\cup N(y)|+d(x_{i+1}) \ge |V(G)|+S-|x, y, x_{i+1}| \ge n$, where S is the number of common neighbors of x_{i+1} and any of $\{x, y\}$ in C_m. Thus, the number of common neighbors of x_{i+1} and any of $\{x, y\}$ is at least 3. Let T be the vertex subset of $N^+_{C_m}(x) \cap N^+_{C_m}(y)$ that is adjacent to x_{i+1}. Clearly, $|T| \ge 3$ (otherwise, this does not satisfy the assumption of the lemma that $|N(x) \cup N(y)| + d(x_{i+1}) \ge n$). Without loss of generality, assume that $|T \cap N^+_{C_m}(x)| \ge 2$. Using $|N_{C_m}(y) \setminus \{x_i\}| \ge 1$, we can construct a C_{m+1} consisting of x and C_m satisfying that there exists some $x_j \in N_{C_m}(y)$ such that path $x_{j-1}x_jx_{j+1}$ of C_m is also a path of C_{m+1}, by Proposition 2.4, we have C_{m+2}. (For example, let $x_{h+1}, x_{k+1} \in T \cap N^+_{C_m}(x)$; if x_h or x_{h+1} is adjacent to y, we construct $C_{m+1} = xx_kx_{k-1}\cdots x_{i+1}x_{k+1}x_{k+2}\cdots x_ix$ consisting of x and C_m satisfying that the paths $x_{h-1}x_hx_{h+1}$ and $x_hx_{h+1}x_{h+2}$ of C_m all also are paths of C_{m+1}. If x_k or x_{k+1} is adjacent to y, we construct $C_{m+1} = xx_hx_{h-1}\cdots x_{i+1}x_{h+1}x_{h+2}\cdots x_ix$ consisting of x and C_m satisfying that the paths $x_{k-1}x_kx_{k+1}$ and $x_kx_{k+1}x_{k+2}$ of C_m all also are paths of C_{m+1}. If y is not adjacent to any of $\{x_h, x_{h+1}, x_k, x_{k+1}\}$, we construct a C_{m+1} consisting of x and C_m and it must be satisfied that path $x_{j-1}x_jx_{j+1}$ of C_m is also a path of C_{m+1} for any $x_j \in N_{C_m}(y) \setminus \{x_i\}$, by Proposition 2.5, we obtain C_{m+2}.)

(2) $N_{C_m}(x) \cap N_{C_m}(y) = \emptyset$. Since there exist $x_{h+1}, x_{k+1} \in N^+_{C_m}(x)$ with $x_{h+1}x_{k+1} \in E(G)$, we can construct $C_{m+1} =: xx_kx_{k-1}\cdots x_{h+1}x_{k+1}x_{k+2}\cdots x_hx$ consisting of x and C_m. (2–1) If y is not adjacent to x_{h+1} or x_{k+1}, using $|N_{C_m}(y)| \ge 2$ and the fact that y is not adjacent to x_h and x_k, there must exist some $x_j \in N_{C_m}(y)$ such that path $x_{j-1}x_jx_{j+1}$ of C_m is also a path of C_{m+1}, by Proposition 2.5, we obtain C_{m+2}. (2–2) If y is adjacent to x_{h+1} and x_{k+1}, then it is easy to obtain a $C_{m+2} =: xx_kx_{k-1}\cdots x_{h+1}yx_{k+1}x_{k+2}\cdots x_hx$ consisting of x, y and C_m.

Case 2. There exists at most a vertex x in $G - C_m$ such that $|N_{C_m}(x)| \ge 2$.

In this case, if G does not have any C_{m+2}, we claim that if $y \in V(G - C_m)$ and $N_{C_m}(y) = \{x_i\}$, then $G[C_m - x_i]$ is complete subgraph of order $m - 1$.

That is, since G is 2-connected, there must exist at least a vertex $y \in V(G-C_m)$ such that $|N_{C_m}(y)| = 1$. Let $N_{C_m}(y) = \{x_i\}$. Then we first prove that $x_{i-1}x_{i+1} \in E(G)$; otherwise, if $x_{i-1}x_{i+1} \notin E(G)$, since not C_{m+2}, if $w \in N[y] \setminus \{x_i\}$ then w is not adjacent to x_{i+1}, x_{i-1}, and hence we can check that $|N(x_{i+1}) \cup N(x_{i-1})| \le n - |N[y] \setminus \{x_i\}| - |\{x_{i+1}, x_{i-1}\}|$, a contradiction.

Then, similarly, we have $x_{i-1}x_{i+2} \in E(G)$; otherwise, if $x_{i-1}x_{i+2} \notin E(G)$, since not C_{m+2}, we can see if $w \in N[y] \setminus \{x_i\}$; then w is not adjacent to x_{i+2}, x_{i-1} (for example, if $wx_{i+2} \in E(G)$, we have $C_{m+2} = x_{i-1}x_{i+1}x_iywx_{i+2}x_{i+3}\cdots x_{i-1}$). Hence we have $|N(x_{i+2}) \cup N(x_{i-1})| \le n - |N[y] \setminus \{x_i\}| - |\{x_{i+2}, x_{i-1}\}|$, a contradiction.

We use induction, under the assumption $x_{i-1}x_{i+r} \in E(G)$, then similarly we have $x_{i-1}x_{i+r+1} \in E(G)$.

Thus, $x_{i+1}, x_{i+2}, x_{i+3}, \ldots, x_{i-3}, x_{i-2}$ are all adjacent to x_{i-1}. Then, clearly, for each pair x_h, x_k in $C_m \setminus \{x_i\}$, $d(x_h, x_k) \le 2$. If $x_hx_k \notin E(G)$. Since, clearly, if $w \in N[y] \setminus \{x_i\}$,n w is not adjacent to x_h, x_k (for example, $wx_h \in E(G)$, together with $x_{i-1}x_{h-1} \in E(G)$, so we have $C_{m+2} = x_{i-1}x_{h-1}x_{h-2}\cdots x_iywx_hx_{h+1}\cdots x_{i-1}$). Hence, we have $|N(x_h)\cup N(x_k)| \le n-|N[y]\setminus\{x_i\}|-|\{x_h, x_k\}|$, a contradiction.

Therefore, $G[V(C_m) \setminus \{x_i\}]$ is a complete subgraph of order $m - 1$.

Then let $P = y_1y_2\cdots y_k$ be a path of $G-C_m$ whose two end-vertices y_1, y_k are adjacent to two vertices x_i, x_j of C_m with the order k of path P being as small as possible. Without loss of generality, assume that $N_{C_m}(y_1) = \{x_i\}$, so $G[C_m - x_i]$ contains a complete subgraph of order $m - 1$.

Subcase 2.1. $k = 2$.

In this case, since $G[C_m - x_i]$ is a complete subgraph of order $m - 1$ and $x_{i-1}x_{j-1} \in E(G)$ (possibly $x_{j-1} = x_i$), we have $C_{m+2} = x_{i-1}x_{j-1}x_{j-2}\cdots x_iy_1y_2x_jx_{j+1}\cdots x_{i-1}$, a contradiction.

Subcase 2.2. $k = 3$.

In this case, since $m \ge 3$, $\max\{|\{x_i, x_{i+1}, \ldots, x_j\}|, |\{x_j, x_{j+1}, \ldots x_i\}|\} \ge 3$. If $\max\{|\{x_i, x_{i+1}, \ldots x_j\}|, |\{x_j, x_{j+1}, \ldots x_i\}|\} = 3$, without loss of generality, assume that $|\{x_i, x_{i+1}, \ldots x_j\}| = 3$; then cycle C_{m+2} consists of path $C_m \setminus \{x_{i+1}\}$ and path $y_1y_2y_3$, where $V(C_{m+2}) = (V(C_m)\cup\{y_1, y_2, y_3\})\setminus\{x_{i+1}\}$, a contradiction. If $\max\{|\{x_i, x_{i+1}, \ldots x_j\}|, |\{x_j, x_{j+1}, \ldots x_i\}|\} \ge 4$, without loss of generality, assume that $|\{x_i, x_{i+1}, \ldots x_j\}| \ge 4$; since x_{j-2} is adjacent to x_{i-1} (possibly $x_{i-1} = x_j$), we can obtain $C_{m+2} = x_jx_{j+1}\cdots x_{i-1}x_{j-2}x_{j-3}x_iy_1y_2y_3x_j$ consisting of vertex set $V(C_m) \setminus \{x_{j-1}\}$ and path $y_1y_2y_3$, where $V(C_{m+2}) = (V(C_m) \cup \{y_1, y_2, y_3\}) \setminus \{x_{j-1}\}$, a contradiction.

Subcase 2.3. $k \ge 4$.

Let x_i, x_j in C_m be adjacent to y_1, y_k, respectively, and $|N_{C_m}(y_1)| = |\{x_i\}|$. Let $x_t \in V(C_m) \setminus \{x_i, x_j\}$, so for any $w \in N[x_t]\setminus\{x_i, y_4\}$, when $w \notin V(C_m)$, since $k \ge 4$, then w is not adjacent to y_1, y_3. Otherwise, if w is adjacent to y_1, in this case we have path $P = wy_1$ of order 2 of $G - C_m$ whose two end-vertices w, y_1 are adjacent to two vertices of C_m, respectively; this contradicts our choice that k is as small as possible. When w is adjacent to y_3, then $wy_3y_4\cdots y_k$ is a path of order less than k of $G-C_m$ whose two end-vertices w, y_k are adjacent to two vertices of C_m, respectively, a contradiction. When $w \in V(C_m)\setminus\{x_i\}$, w is not adjacent to y_1, y_3. This is because, since $N_{C_m}(y_1) = \{x_i\}$, $wy_1 \notin E(G)$. If $wy_3 \in E(G)$, then $y_1y_2y_3$ is a path of order less than k of $G - C_m$ whose two end-vertices y_1, y_3 are adjacent to two vertices x_i, w of C_m, respectively, a contradiction. Hence we have $|N(y_1) \cup N(y_3)| \le n - |N[x_t] \setminus \{x_i, y_4\}| - |\{y_1, y_3\}|$, a contradiction. $\square$

Therefore, this completes the proof of Lemma 2.6.

Note. A pancyclic graph is an important subject in graph theory and related areas. Recently, some related interesting works on pancyclicity have been published in [4,5,8,11,6,10,9,7,15]; among them, many have been widely used in computer science as well as information science.

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