Suggestions for the authors:

For the title, I suggest something like 'A weak Ore-type condition for pancyclicity involving vertices at distance two'.

The purpose of an abstract is to tell the reader what is done in the paper. This is not the place to include the history of the subject. I suggest something along the following lines.

> **Abstract.** Let G be a graph of order n. It is proved that if $d(x) + d(y) \geq n - 1$ for every pair of vertices x, y of G that are distance two apart, then G is pancyclic or $G \in \{G_{(n-1)/2} \vee \bar{K}_{(n+1)/2}, K_{n/2,n/2}, K_{n/2,n/2} - e, C_5\}$. This strengthens known results of Ainouche and Christofides and of Aldred, Holton and Zhang, in which the degree-sum condition was imposed on all pairs of nonadjacent vertices, not just those at distance two.

From the Abstract up to Theorem 1.5 you write $K^C_{(n+1)/2} \vee G_{(n-1)/2}$, and you then switch to writing $G_{(n-1)/2} \vee K^C_{(n+1)/2}$. It doesn't matter which order you use, but please be consistent. It is more usual to represent the complement of G by $\bar{G}$; if you want to use a superscript C, then it should be in roman type.

The string of definitions in the opening paragraph should be kept as short as possible. Notation that is not used until Section 2 can be defined at the start of Section 2. It seems that δ is used only once, unnecessarily, in Theorem 1.1, and C^n_m*-pancyclic* is not used at all; if so, they should not be defined. However, the join is used on page 2 and not defined until page 3, and $d(x, y)$ is not defined at all; they should be defined before they are used.

I suggest you amalgamate Theorems 1.6 and 1.7, but also interchange them, so that in each pair of results with the same hypotheses, the one about hamiltonicity always precedes the one about pancyclicity.

Corollary 1.8 already follows from Theorem 1.5. If you want to include it, you should say this; but I would suggest that you delete it.

At the end of the paper, '**Note that 1:**' should be '**Note 1.**' However, your use of $\vee$ here is non-standard, and you do not explain what condition G_m^* must satisfy. If you are keen to include this note, you should explain it more carefully, and include it in Section 1; otherwise you should delete it. Note 2 is rather trivial, but it is easily included in Section 1.

The order in Section 2 seems very strange. The proof of Theorem 1.7 is quite straightforward, from first principles, so why is it sandwiched between Lemma 2.2 and Lemma 2.3? It would be better to prove it first, then prove the lemmas, and then make your observation that Theorem 1.6 follows easily from the lemmas and Theorem 1.7.

I cannot see anywhere that Lemma 2.1 is used. If it is not used, then you should not include it. If you do include it, it would be better to label its parts (a), (b), etc., rather than (1), (2), etc., so that there is no possible confusion with equations labelled (1), (2).

Lemma 2.4 is wrong (it omits $K_{2,3}$ and $K_2 \vee \bar{K}_3$), but it is also unnecessary; you can say all that needs saying in one sentence at the start of the proof of Theorem 1.6.

It is better not to use C_m to denote both a generic m-cycle and a specific m-cycle used in the proof. Calling the latter C would also avoid many double subscripts.

(In the proof of Theorem 1.7) the statement 'v is not adjacent to x_{i+1} or x_{j+1}' means that v is not adjacent to x_{i+1} AND v is not adjacent to x_{j+1}. What you mean is that v is not adjacent to both x_{i+1} and x_{j+1} (but it may be adjacent to one of them). This needs careful rewording.

Here is my suggestion for the first part of the paper.

1 Introduction

We consider a finite undirected simple graph G with vertex-set $E(G)$ and edge-set $V(G)$. If $x, y \in V(G)$ then $d(x)$ and $d(x, y)$ denote the degree of x and the distance between x and y, respectively. If G and H are graphs, then $\bar{G}$ is the complement of G, and $G \vee H$ is the join of disjoint copies of G and H. The complete graph and cycle of order n are denoted by K_n and C_n, and $K_{p,q} = \bar{K}_p \vee \bar{K}_q$ is a complete bipartite graph. We write $G \in G_{(n-1)/2} \vee \bar{K}_{(n+1)/2}$ if G is the join of $\bar{K}_{(n+1)/2}$ and any graph of order $(n-1)/2$.

A graph of order n is *pancyclic* if it contains cycles of every length k, $3 \le k \le n$. Other terminology and notation can be found in [4].

The following results are well known.

Theorem 1.1. (Dirac [5]) *If G is a graph of order n with minimum degree at least $n/2$, then G is Hamiltonian.*

Theorem 1.2. (Ore [7]) *If G is a graph of order $n \geq 3$, and $d(x) + d(y) \geq n$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G is Hamiltonian.*

Theorem 1.3. (Bondy [3]) *If G is a graph of order $n \geq 3$, and $d(x) + d(y) \geq n$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G is pancyclic or $G = K_{n/2,n/2}$.*

The next two theorems are strengthenings of Theorems 1.2 and 1.3 in which Ore's condition is weakened by one. The condition of 2-connectedness is needed in order to rule out graphs of the form $K_1 \vee (K_r \cup K_{n-1-r})$.

Theorem 1.4. (Ainouche and Christofides [1]) *If G is a 2-connected graph of order $n \geq 3$, and $d(x) + d(y) \geq n - 1$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G is Hamiltonian or $G \in G_{(n-1)/2} \vee \bar{K}_{(n+1)/2}$.*

Theorem 1.5. (Aldred, Holton and Zhang [2]) *If G is a 2-connected graph of order $n \geq 3$, and $d(x) + d(y) \geq n - 1$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G is pancyclic or $G \in \{G_{(n-1)/2} \vee \bar{K}_{(n+1)/2}, K_{n/2,n/2}, C_5\}$.*

In this paper we prove the following result, which strengthens Theorems 1.4 and 1.5 by imposing the degree-sum condition only on pairs of vertices at distance two.

Theorem 1.6. *Let G be a 2-connected graph of order $n \geq 3$, in which $d(x) + d(y) \geq n - 1$ for each pair of vertices $x, y \in V(G)$ such that $d(x, y) = 2$. Then the following hold.*
(a) *G is Hamiltonian or $G \in G_{(n-1)/2} \vee \bar{K}_{(n+1)/2}$.*
(b) *G is pancyclic or $G \in \{G_{(n-1)/2} \vee \bar{K}_{(n+1)/2}, K_{n/2,n/2}, K_{n/2,n/2} - e, C_5\}$.*

Clearly, Theorem 1.6(a) implies Theorem 1.4 and Theorem 1.6(b) implies Theorem 1.5.

2 Proof of Theorem 1.6

We will use the following notation. If H and S are subsets of $V(G)$ or subgraphs of G, we denote by $N_H(S)$ the set of vertices in H which are adjacent to some vertex in S, and set $d_H(S) = |N_H(S)|$. If $u \in V(G)$ then we shorten $N_G(\{u\})$ to $N(u)$. We denote by $G - H$ [and $G[S]$? – ever used?] the induced subgraphs of G on $V(G) \setminus V(H)$ [and S?], respectively. If $C = x_1 x_2 \ldots x_m x_1$ is a cycle of order m, we write $N_C^+(u) = \{x_{i+1} : x_i \in N_C(u)\}$, $N_C^-(u) = \{x_{i-1} : x_i \in N_C(u)\}$, and $N_C^{\pm}(u) = N_C^+(u) \cup N_C^-(u)$, where subscripts are taken modulo m.

Let G be a graph of order n satisfying the hypotheses of Theorem 1.6.

Proof of Theorem 1.6(a). Assume that G is not Hamiltonian, and let $C = x_1x_2 \dots x_mx_1$ be a longest cycle of G. Choose vertices $u \in V(G-C)$ and $x_i \in V(C)$ such that $ux_i \in E(G)$, let H be the component of $G-C$ containing u, and let $x_j \in N_C(H) \setminus \{x_i\}$, which exists since G is 2-connected.

Claim 1. *u is adjacent to every vertex in $V(G-C) \setminus \{u\}$, so that $H = G - C$.*

Proof. Let P be a path in H connecting u to a neighbor of x_j. (Possibly $V(P) = \{u\}$.) Clearly no vertex of H is adjacent to x_{i+1}, as this would give a longer cycle than C. Suppose some vertex $v \in V(G-C) \setminus \{u\}$ is not adjacent to u.

If v is adjacent to both x_{i+1} and x_{j+1}, then $v \notin V(P)$ since no vertex of H is adjacent to x_{i+1}, and so $x_iPx_jx_{j-1} \dots x_{i+1}vx_{j+1}x_{j+2} \dots x_i$ is a longer cycle than C. This is a contradiction, and so w.l.o.g. we may assume that v is not adjacent to x_{i+1}. Since C is a longest cycle, it is easy to see that x_{i+1} is not adjacent to any vertex in $N_{G-C}(u)$ or $N_C^+(u)$ (that is, if $ux_h \in E(G)$ then $x_{i+1}x_{h+1} \notin E(G)$). Thus

$$N(x_{i+1}) \subseteq V(G) \setminus (N_C^+(u) \cup N_{G-C}(u) \cup \{u, v\}),$$

so that $d(x_{i+1}) + d(u) \leq n - 2$. But this contradicts the degree-sum hypothesis of Theorem 1.6, since clearly $d(u, x_{i+1}) = 2$; and this contradiction proves Claim 1. □

Claim 2. *$m = n - 1$ and $V(G-C) = \{u\}$, and u has a neighbor $x_k \in V(C) \setminus \{x_i\}$ such that $d(x_{i+1}) + d(x_{k+1}) = n - 1$.*

Proof. Recall that x_{i+1} has no neighbor in H, and $H = G - C$ by Claim 1. Since $d(u, x_{i+1}) = 2$, and vertices u, x_{i+1}, x_{j+1} are not adjacent to either u or x_{i+1}, it follows from the degree-sum hypothesis of the theorem that there is a vertex $x_k \in V(C)$ $(k \neq i)$ such that x_k is adjacent to both u and x_{i+1}, so that $d(x_{i+1}, x_{k+1}) = 2$.

We now use a standard argument. Let $A = \{x_{i+2}, x_{i+3}, \dots, x_k\}$ and $B = \{x_{k+2}, x_{k+3}, \dots, x_i\}$. If $x_h \in A \cap N(x_{i+1})$ then $x_{h-1} \notin N(x_{k+1})$, since otherwise we easily find a longer cycle than C. Thus the number of edges between $\{x_{i+1}, x_{k+1}\}$ and A is at most $|A| + 1$. Similarly, the number of edges between $\{x_{i+1}, x_{k+1}\}$ and B is at most $|B| + 1$. Since there are no edges between $\{x_{i+1}, x_{k+1}\}$ and any vertex in the set $V(H) \cup \{x_{i+1}, x_{k+1}\}$, it follows from the hypothesis of the theorem that

$$n - 1 \leq d(x_{i+1}) + d(x_{k+1}) \leq |A| + 1 + |B| + 1 = |V(C)| = m.$$

Since $m < n$, it follows that equality holds throughout, which proves Claim 2. □

Claim 3. $d(u) = (n-1)/2$.

Proof. Since each two of u, x_{i+1}, x_{k+1} are at distance two, the hypothesis of the theorem implies that $d(u) + d(x_{i+1}) \geq n - 1$ and $d(u) + d(x_{k+1}) \geq n - 1$, which by Claim 2 implies that $d(u) \geq (n-1)/2$. But if $d(u) > (n-1)/2$ then we can insert u into C to give a longer cycle, and this contradiction proves Claim 3. $\square$

Let $X = N_C^+(u) \cup \{u\}$ and $Y = V(G) \setminus X$. It follows from Claim 3 that the neighbors of u are alternate vertices of C, and that X is an independent set of $(n+1)/2$ vertices, each of which, other than u, is at distance two from u. Thus $N(X) \subseteq Y$, so that each vertex in X has degree at most $|Y| = (n-1)/2$. By Claim 3, for the degree-sum hypothesis to hold, each vertex in X must have degree $(n-1)/2$ and be adjacent to all vertices in Y, which means that $G \in G_{(n-1)/2} \vee \bar{K}_{(n+1)/2}$. This completes the proof of Theorem 1.6(a). $\square$

We now turn to the proof of Theorem 1.6(b). Since G is 2-connected, every vertex has degree at least 2, and so if $n \leq 5$ then the result follows from Theorem 1.5. Thus we may assume that $n \geq 6$. We need two lemmas.

Lemma 2.1. *If G contains a cycle of length $m \geq 5$, then G contains a cycle of length $m - 2$.*

Lemma 2.2. *If G contains no cycle of length $n-1$, then $G \in \{K_{n/2,n/2}, K_{n/2,n/2} - e\}$.*