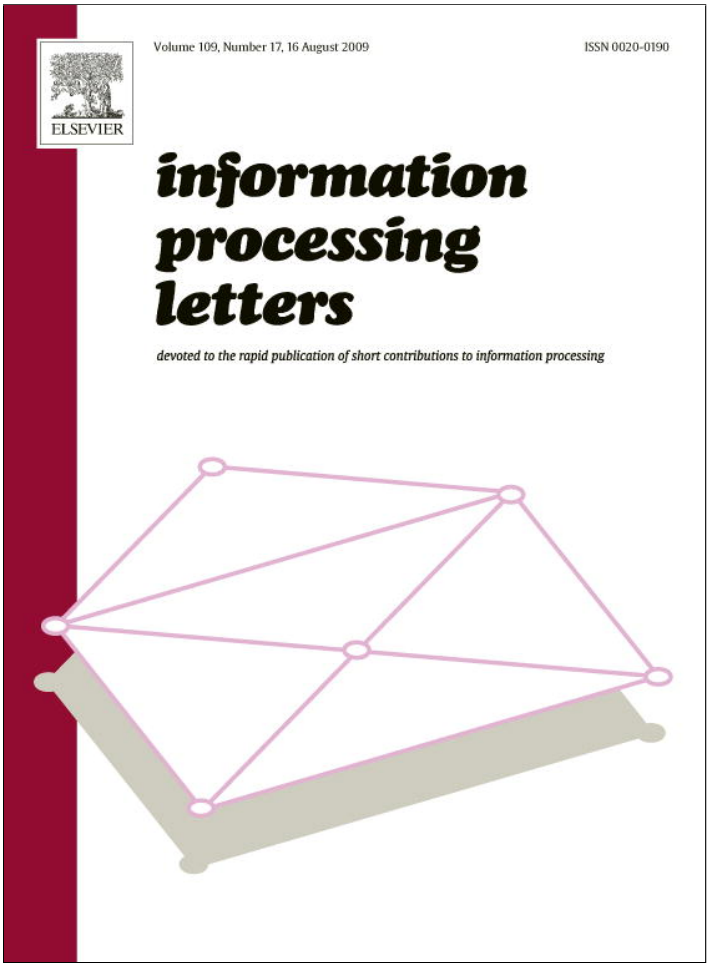
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A sufficient condition for pancyclic graphs

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ABSTRACT

In 2005, Rahman and Kaykobad proved that if G is a connected graph of order n such that $d(x)+d(y)+d(x,y) \geqslant n+1$ for each pair x, y of distinct nonadjacent vertices in G, where $d(x,y)$ is the length of a shortest path between x and y in G, then G has a Hamiltonian path [Inform. Process. Lett. 94 (2005) 37–41]. In 2006 Li proved that if G is a 2-connected graph of order $n \geqslant 3$ such that $d(x)+d(y)+d(x,y) \geqslant n+2$ for each pair x, y of nonadjacent vertices in G, then G is pancyclic or $G = K_{n/2,n/2}$ where $n \geqslant 4$ is an even integer [Inform. Process. Lett. 98 (2006) 159–161]. In this work we prove that if G is a 2-connected graph of order n such that $d(x)+d(y)+d(x,y) \geqslant n+1$ for all pairs x, y of distinct nonadjacent vertices in G, then G is pancyclic or G belongs to one of four specified families of graphs.

1. Introduction

We consider only simple graphs, i.e., graphs with no multi-edges and no self loops, and every reference to a cycle or a path, unless otherwise specified, indicates, respectively, a simple cycle or a simple path. For a graph G, let $V(G)$ be the vertex set of G and $E(G)$ the edge set of G. The complete graph of order n is denoted by K_n, and the complete bipartite graph with the partite sets A and B with $|A| = p$ and $|B| = q$ is denoted by $K_{p,q}$. For two vertices u and v, let $d(u,v)$ be the length of a shortest path between vertices u and v in G, that is, $d(u,v)$ is the *distance* between u and v. The *minimum degree* of a graph G is denoted by $\delta(G)$ (or δ if the graph G under consideration is understood). For a subgraph H of a graph G and a subset S of $V(G)$, let $N_H(S)$ be the set of vertices in H that are adjacent to some vertex in S and let the cardinality of $N_H(S)$ be $|N_H(S)| = d_H(S)$. In particular, if $H = G$ and $S = \{u\}$, then let $N_G(S) = N(u)$, which is the *neighborhood* of u in G. In this case, the cardinality of $N_G(S)$ is denoted by $d_G(S) = |N(u)| = d(u)$, which is the *degree* of u.

Furthermore, let $G - H$ and $G[S]$ denote the subgraphs of G induced by $V(G) - V(H)$ and S, respectively. For each integer $m \geqslant 3$, let

$$C_m = x_1 x_2 \ldots x_m x_1$$

denote a cycle of length m and define

$$N^+_{C_m}(u) = \{x_{i+1}\colon x_i \in N_{C_m}(u)\},$$
$$N^-_{C_m}(u) = \{x_{i-1}\colon x_i \in N_{C_m}(u)\},$$
$$N^\pm_{C_m}(u) = N^+_{C_m}(u) \cup N^-_{C_m}(u),$$

where subscripts are expressed as integers modulo m.

A cycle in a graph G that contains every vertex of G is called a *Hamiltonian cycle* of G. A *Hamiltonian graph* is a graph that contains a Hamiltonian cycle. A path in a graph G that contains every vertex of G is called a *Hamiltonian path* in G. A graph G is said to be *r-pancyclic* if G contains a cycle of length k for each k with $r \leqslant k \leqslant n$. A *3-pancyclic* graph is simply called a *pancyclic graph*. We refer to the book [1] for graph theory notation and terminology not described in this paper.

It is well known that the Hamiltonian graph problem is NP-complete [2]. In 2005, Rahman and Kaykobad [5] obtained the following result:

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Theorem 1.1. *(See Rahman and Kaykobad, 2005 [5].) If G is a connected graph of order $n \geqslant 3$ such that $d(x) + d(y) + d(x, y) \geqslant n + 1$ for each pair x, y of nonadjacent vertices in G, then G has a Hamiltonian path.*

In 2006, Li [3] considered other Hamiltonian properties of graphs under same or similar conditions as Theorem 1.1. In order to present results obtained by Li in [3], we first introduce some additional definitions. For two graphs F and H the *join* $F \vee H$ of F and H is the graph constructed from F and H by adding all edges joining the vertices of F and the vertices of H. In [3] two classes $\mathcal{C}_n$ and $\mathcal{D}_n$ of graphs of order n are defined as follows. A graph G of order n belongs to the family $\mathcal{C}_n$ if the vertex set of G is $V(G) = V(K_1) \cup V(G_1) \cup V(G_2)$, where K_1 is a trivial graph, $G_1 = K_{p_1} \vee K_{q_1} = K_{p_1+q_1}$ is the complete graph of order $p_1 + q_1$ with $p_1 \geqslant 1$ and $q_1 \geqslant 0$, $G_2 = K_{p_2} \vee K_{q_2}$ is the complete graph of order $p_2 + q_2$, $p_2 \geqslant 1$ and $q_2 \geqslant 0$, and $V(K_1)$, $V(G_1)$, and $V(G_2)$ are pairwise disjoint sets with $n = p_1 + q_1 + p_2 + q_2 + 1$, and the edge set of G is

$$E(G) = E(G_1) \cup E(G_2) \cup \left\{ab\colon\ a \in V(K_1), b \in V(K_{p_1}) \cup V(K_{p_2})\right\}.$$

The family $\mathcal{D}_n$ of graphs of order n is defined as

$$\left\{G\colon\ K_{p,p+1} \subseteq G \subseteq K_p \vee (p+1)K_1,\ \left|V(G)\right| = 2p + 1 = n \geqslant 3\right\},$$

where $(p + 1)K_1$ is the complement of K_{p+1}, that is, the empty graph of order $p + 1$.

Theorem 1.2. *(See Li, 2006 [3].) Let G be a connected graph of order $n \geqslant 6$. If $d(x) + d(y) + d(x, y) \geqslant n + 1$ for each pair x, y of nonadjacent vertices of G, then G is Hamiltonian or $G \in \mathcal{C}_n \cup \mathcal{D}_n$.*

Theorem 1.3. *(See Li, 2006 [3].) If G is a 2-connected graph of order $n \geqslant 3$ such that $d(x) + d(y) + d(x, y) \geqslant n + 2$ for each pair x, y of nonadjacent vertices in G, then G is pancyclic or $G = K_{n/2,n/2}$ where $n \geqslant 4$ is an even integer.*

In this paper, we present the following result, which improves Theorem 1.3.

Theorem 1.4. *Let G be a 2-connected graph of order $n \geqslant 6$. If $d(x) + d(y) + d(x, y) \geqslant n + 1$ for all pairs x, y of nonadjacent vertices in G, then either G is pancyclic or*

$$G \in \mathcal{C}_n \cup \mathcal{D}_n \cup \{K_{n/2,n/2}, K_{n/2,n/2} - e\colon\ n \text{ is even}\}.$$

2. Proofs of main results

The proof of Theorem 1.4 is based on the following lemmas or theorem.

Lemma 2.1. *Let $C_m = x_1x_2 \ldots x_mx_1$ be a cycle length m of a graph G and let $v \in V(C_m)$. If there does not exist C_{m-2} containing v in G, then, for each integer i with $1 \leqslant i \leqslant m$, the following hold.*

(1) *If $v \notin \{x_{i+1}, x_{i+2}\}$, then $x_ix_{i+3} \notin E(G)$.*

(2) *If $v \notin \{x_{i+1}, x_{i+2}, x_{i+3}\}$, then $N_{G-C_m}(x_i) \cap N_{G-C_m}(x_{i+4}) = \emptyset$.*

(3) *If $v \neq x_{i+1}$ and $x_ix_{i+2} \in E(G)$, then when $v \neq x_{j+1}$ we have $x_jx_{j+2} \notin E(G)$ and when $v \notin \{x_{j+1}, x_{j+2}\}$ we have $N_{G-C_m}(x_j) \cap N_{G-C_m}(x_{j+3}) = \emptyset$ for any $j \neq i, i+1$. Similarly, if $v \notin \{x_{i+1}, x_{i+2}\}$ and $N_{G-C_m}(x_i) \cap N_{G-C_m}(x_{i+3}) \neq \emptyset$, then when $v \neq x_{j+1}$ we have $x_jx_{j+2} \notin E(G)$ and when $v \notin \{x_{j+1}, x_{j+2}\}$ we have $N_{G-C_m}(x_j) \cap N_{G-C_m}(x_{j+3}) = \emptyset$ for any $j \neq i, i+1, i+2$.*

(4) *If $x_ix_h \in E(G)$, where $h \neq i+1, i+2$, then when $v \notin \{x_{i+1}, x_{i+2}\}$ we have $x_{i+3}x_{h+1} \notin E(G)$ and when $v \notin \{x_{i+1}, x_{h+1}\}$ we have $x_{i+2}x_{h+2} \notin E(G)$.*

Proof. Let i be an integer with $1 \leqslant i \leqslant m$. Recall that the subscripts of vertices are expressed as integers modulo m.

(1) If $x_ix_{i+3} \in E(G)$, then there exists $C_{m-2} = x_1x_2 \ldots x_ix_{i+3} \ldots x_mx_1$ in G containing v, a contradiction.

(2) If there is $u \in N_{G-C_m}(x_i) \cap N_{G-C_m}(x_{i+4}) \neq \emptyset$, then we get $C_{m-2} = x_1x_2 \ldots x_iux_{i+4} \ldots x_mx_1$ in G containing v, a contradiction.

(3) First suppose that $x_ix_{i+2} \in E(G)$ with $v \neq x_{i+1}$, and $x_jx_{j+2} \in E(G)$ with $v \neq x_{j+1}$ for some $j \neq i, i+1$. Thus if $j > i$, then we obtain $C_{m-2} = x_1x_2 \ldots x_ix_{i+2}x_{i+3} \ldots x_jx_{j+2}x_{j+3} \ldots x_mx_1$ in G containing v; while if $j < i$, we obtain $C_{m-2} = x_1x_2 \ldots x_jx_{j+2}x_{j+3} \ldots x_ix_{i+2}x_{i+3} \ldots x_mx_1$ in G containing v. In each case, a contradiction is produced.

Next, suppose that $x_ix_{i+2} \in E(G)$ with $v \neq x_{i+1}$, and $N_{G-C_m}(x_j) \cap N_{G-C_m}(x_{j+3}) \neq \emptyset$ with $v \notin \{x_{j+1}, x_{j+2}\}$ for some $j \neq i, i+1$. Let $u \in N_{G-C_m}(x_j) \cap N_{G-C_m}(x_{j+3})$. Thus if $j > i$, then we obtain $C_{m-2} = x_1x_2 \ldots x_ix_{i+2}x_{i+3} \ldots x_jux_{j+3}x_{j+4} \ldots x_mx_1$ in G containing v; while if $j < i$, then we obtains $C_{m-2} = x_1x_2 \ldots x_jux_{j+3}x_{j+4} \ldots x_ix_{i+2}x_{i+3} \ldots x_mx_1$ in G containing v. Again, a contradiction is produced in each case.

Now suppose that $u \in N_{G-C_m}(x_i) \cap N_{G-C_m}(x_{i+3}) \neq \emptyset$ with $v \notin \{x_{i+1}, x_{i+2}\}$, and $x_jx_{j+2} \in E(G)$ with $v \neq x_{j+1}$ for some $j \neq i, i+1, i+2$. Thus if $j > i$, then we obtain $C_{m-2} = x_1x_2 \ldots x_iux_{i+2}x_{i+3} \ldots x_jx_{j+2}x_{j+3} \ldots x_mx_1$ in G containing v; while if $j < i$, then we obtain $C_{m-2} = x_1x_2 \ldots x_jx_{j+2}x_{j+3} \ldots x_iux_{i+2}x_{i+3}x_mx_1$ in G containing v. A contradiction is produced in each case.

Finally, suppose that $u \in N_{G-C_m}(x_i) \cap N_{G-C_m}(x_{i+3}) \neq \emptyset$ with $v \notin \{x_{i+1}, x_{i+2}\}$, and $w \in N_{G-C_m}(x_j) \cap N_{G-C_m}(x_{j+3}) \neq \emptyset$ with $v \notin \{x_{j+1}, x_{j+2}\}$ for some $j \neq i, i+1, i+2$. Thus if $j > i$, then we obtain $C_{m-2} = x_1x_2 \ldots x_iux_{i+3}x_{i+4} \ldots x_jwx_{j+3}x_{j+4} \ldots x_mx_1$ in G containing v; while if $j < i$, then we obtain $C_{m-2} = x_1x_2 \ldots x_jwx_{j+3}x_{j+4} \ldots x_iux_{i+3}x_{i+4} \ldots x_mx_1$ in G containing v. In each case, a contradiction is produced in each case.

(4) If $x_ix_h \in E(G)$, where $h \neq i+1, i+2$ and $x_{i+3}x_{h+1} \in E(G)$ (say $h \geqslant i$), then we obtain $C_{m-2} = x_1x_2 \ldots x_ix_hx_{h-1} \ldots x_{i+3}x_{h+1}x_{h+2} \ldots x_mx_1$ in G containing v, a contradiction. On the other hand, if $x_ix_h \in E(G)$, where $h \neq i+1$, $i+2$ and $x_{i+2}x_{h+2} \in E(G)$, then we obtain $C_{m-2} = x_1x_2 \ldots x_ix_hx_{h-1} \ldots x_{i+2}x_{h+2}x_{h+3} \ldots x_mx_1$ in G containing v, a contradiction again.

This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let G be a 2-connected graph of order $n \geqslant 6$ such that $d(x) + d(y) \geqslant n - 1$ for each pair x, y of nonadjacent ver-*

tices in G with $d(x, y) = 2$. If there exists an m-cycle C_m, where $m \geqslant 5$, in G, then G contains C_{m-2}.

Proof. Let $C_m = x_1x_2\ldots x_mx_1$. Assume, to the contrary, that there does not exist any C_{m-2} in G. We consider the following four cases, according to the values of m.

Case 1. $m = 5$. Let $C_5 = x_ix_{i+1}x_{i+2}x_{i+3}x_{i+4}x_i$. Since G contains no C_3, it follows that $x_ix_{i+2} \notin E(G)$ for any x_i in C_5 (for otherwise, if $x_ix_{i+2} \in E(G)$, then we obtain $C_3 = x_ix_{i+1}x_{i+2}x_i$, a contradiction). Furthermore, $N(x_i) \cap V(C_5) = \{x_{i-1}, x_{i+1}\}$.

Since $d(x_i) + d(x_{i+2}) \geqslant n - 1$ and $|N(x_i) \cap V(C_5)| = |N(x_{i+2}) \cap V(C_5)| = 2$, it follows that $|N_{G-C_m}(x_i)| + |N_{G-C_m}(x_{i+2})| \geqslant n - 5$.

Subcase 1.1. $|N(x_i) \cap V(G - C_m)| \geqslant |V(G - C_m)|/2 + 1$ *for some vertex x_i of C_m*. By the similar arguments as above, we have

$$\left|N_{G-C_m}(x_{i-1})\right| + \left|N_{G-C_m}(x_{i+1})\right| \geqslant n - 5 = \left|V(G - C_m)\right|.$$

This implies that $|N_{G-C_m}(x_{i-1})| \geqslant |V(G - C_m)|/2$ or $|N_{G-C_m}(x_{i+1})| \geqslant |V(G - C_m)|/2$. Without loss of generality, assume that $|N_{G-C_m}(x_{i-1})| \geqslant |V(G - C_m)|/2$. Since $|N(x_i) \cap V(G - C_m)| \geqslant |V(G - C_m)|/2 + 1$, it follows that

$$\left|N_{G-C_m}(x_{i-1})\right| + \left|N_{G-C_m}(x_i)\right| \geqslant \left|V(G - C_m)\right| + 1,$$

and so there exists $u \in N_{G-C_m}(x_{i-1}) \cap N_{G-C_m}(x_i)$. Therefore, G contains $C_3 = x_iux_{i-1}x_i$, which is a contradiction.

Subcase 1.2. $|N(x_i) \cap V(G - C_m)| \leqslant |V(G - C_m)|/2$ *for each vertex x_i of C_m*. Since $|N_{G-C_m}(x_i)| + |N_{G-C_m}(x_{i+2})| \geqslant n - 5 = |V(G - C_m)|$ in this case and $|N(x_i) \cap V(G - C_m)| \leqslant |V(G - C_m)|/2$ in this subcase, and $|N(x_{i+2}) \cap V(G - C_m)| \leqslant |V(G - C_m)|/2$ for every vertex x_i of C_m, it follows that

$$\begin{aligned}\left|N(x_i) \cap V(G - C_m)\right| &= \left|N(x_{i+2}) \cap V(G - C_m)\right| \\ &= \left|V(G - C_m)\right|/2\end{aligned}$$

for every vertex x_i. Furthermore, there must exist x_j, x_{j+1} in the odd cycle C_5 such that both x_j and x_{j+1} are adjacent to a common vertex u of $G - C_5$. Therefore, there is a 3-cycle $C_3 = x_jux_{j-1}x_j$ in G, which is a contradiction.

Case 2. $m = 6$. Let $C_6 = x_1x_2x_3x_4x_5x_6x_1$.

Subcase 2.1. $x_ix_{i+2} \in E(G)$ *for some vertex x_i in C_6*. Without loss of generality, assume that $x_1x_3 \in E(G)$. By Lemma 2.1(3), we have $x_4x_6 \notin E(G)$, and Lemma 2.1(1), we have $x_3x_6, x_1x_4 \notin E(G)$. Furthermore, we claim that $x_2x_4, x_2x_6 \notin E(G)$ (for otherwise, if $x_2x_4 \in E(G)$, we get $C_4 = x_1x_3x_4x_2x_1$, a contradiction; while if $x_2x_6 \in E(G)$, we get $C_4 = x_1x_3x_2x_6x_1$, a contradiction). Then it can be verified that $|N_{C_m}(x_4)| + |N_{C_m}(x_6)| \leqslant 4$. We also have $N_{G-C_m}(x_4) \cap N_{G-C_m}(x_6) = \emptyset$ (for otherwise, if $u \in N_{G-C_m}(x_4) \cap N_{G-C_m}(x_6)$, then $C_4 = x_4x_5x_6ux_4$, a contradiction). However then, $d(x_4) + d(x_6) \leqslant n - 2$, which contradicts the condition of Lemma 2.2.

Subcase 2.2. $x_ix_{i+2} \notin E(G)$ *for any vertex x_i in C_6*. In this case, it can be verified that $|N_{C_m}(x_4)| + |N_{C_m}(x_6)| \leqslant 4$ (by Lemma 2.1). Using the similar arguments as that used in subcase 2.1, we can show that $N_{G-C_m}(x_4) \cap N_{G-C_m}(x_6) = \emptyset$. Furthermore, we can also apply the similar arguments as subcase 2.1 to obtain $d(x_4) + d(x_6) \leqslant n - 2$, which contradicts the condition of Lemma 2.2.

Case 3. $m = 7$. Let $C_7 = x_1x_2x_3x_4x_5x_6x_7x_1$.

Subcase 3.1. $x_ix_{i+2} \in E(G)$ *for some vertex x_i in C_7*. Without loss of generality, assume that $x_1x_3 \in E(G)$. Then we have $x_2x_4 \notin E(G)$ or $x_7x_2 \notin E(G)$ (for otherwise, if $x_2x_4, x_7x_2 \in E(G)$, then we obtain C_5: $x_2x_4x_5x_6x_7x_2$, a contradiction). We may also assume that $x_2x_4 \notin E(G$. Then x_2 can only be adjacent to x_3, x_7, x_1 of $V(C_7)$. (For otherwise, we will obtain a C_5, a contradiction. For example, if x_2 is adjacent to x_5, then we obtain a C_5: $x_2x_5x_6x_7x_1x_2$, a contradiction.) Moreover, x_2 and x_4 do not have any common neighbor in $G - C_7$ (for otherwise, if $v \in V(G - C_m)$ is adjacent to x_2 and x_4, then we obtain a C_5: $x_2vx_4x_3x_1x_2$, a contradiction). Thus, it can be verified that $d(x_2) + d(x_4) \leqslant n - 2$, which contradicts the condition of Lemma 2.2.

Subcase 3.2. $x_ix_{i+2} \notin E(G)$ *for any vertex x_i in C_7*. In this case, x_2 and x_5 do not have any common neighbor in $G - C_7$, for otherwise, there is a 5-cycle C_5 in G. Thus, we have

$$\min\left\{d_{G-C_7}(x_2), d_{G-C_7}(x_5)\right\} \leqslant \left|V(G - C_7)\right|/2.$$

Assume, without loss of generality, that $d_{G-C_7}(x_5) \leqslant |V(G - C_7)|/2$. Similarly, x_3 and x_7 do not have any common neighbor in $G - C_7$ (for otherwise, there is a C_5 in G). Thus,

$$\min\left\{d_{G-C_7}(x_3), d_{G-C_7}(x_7)\right\} \leqslant \left|V(G - C_7)\right|/2.$$

We may assume, without loss of generality, that $d_{G-C_7}(x_3) \leqslant |V(G - C_7)|/2$. It then can be verified that $d_{C_m}(x_3) + d_{C_m}(x_5) = |\{x_2, x_4, x_6\}|$. We then obtain $d(x_3) + d(x_5) \leqslant n - 2$, which contradicts the condition of Lemma 2.2.

Case 4. $m \geqslant 8$. Let $C_m = x_1x_2x_3x_4\ldots x_mx_1$.

Subcase 4.1. $x_ix_{i+2} \in E(G)$ *for some vertex x_i in C_m*. Without loss of generality, assume that $x_1x_3 \in E(G)$. In this case, we have $x_2x_4 \notin E(G)$ or $x_2x_m \notin E(G)$ (for otherwise, if $x_2x_4, x_2x_m \in E(G)$, then we can obtain C_{m-2}: $x_2x_4x_5\ldots x_mx_2$, a contradiction). Again, we may assume, without loss of generality, that $x_2x_4 \notin E(G)$. Then x_4 and x_8 do not have any common neighbor in $G - C_m$ (for otherwise, if $v \in V(G - C_m)$ is adjacent to x_4 and x_8, then we also obtain a C_{m-2}: $x_4vx_8x_9\ldots x_4$, a contradiction). Thus, we have

$$\min\left\{d_{G-C_m}(x_4), d_{G-C_m}(x_8)\right\} \leqslant \left|V(G - C_m)\right|/2.$$

Without loss of generality, assume that $d_{G-C_m}(x_4) \leqslant |V(G - C_m)|/2$. Similarly, we have x_2 and x_6 do not have any common neighbor in $G - C_m$. Thus,

$$\min\left\{d_{G-C_m}(x_2), d_{G-C_m}(x_6)\right\} \leqslant \left|V(G - C_m)\right|/2.$$

We may assume that $d_{G-C_m}(x_2) \leqslant |V(G-C_m)|/2$. Thus

$$d_{G-C_m}(x_2) + d_{G-C_m}(x_4) \leqslant \left|V(G-C_m)\right|/2 + \left|V(G-C_m)\right|/2 = \left|V(G-C_m)\right| \quad (1)$$

If $x_r \in \{x_1, x_2, \ldots, x_m\} \setminus \{x_1, x_3\}$ is adjacent to x_2, then x_{r+2} is not adjacent to x_4 (for otherwise, we obtain a C_{m-2}: $x_2 x_r x_{r-1} \ldots x_4 x_{r+2} x_{r+3} \ldots x_2$, a contradiction). Clearly x_2, x_4, x_5, x_{m-1} are not adjacent to x_2, and x_4, x_6, x_7, x_1 are not adjacent to x_4, respectively. Hence we have

$$d_{C_m}(x_4) \leqslant m - \left|N_{C_m}(x_2) \setminus \{x_1, x_3\}\right| - \left|\{x_4, x_6, x_7, x_1\}\right| \leqslant m - d_{C_m}(x_2) - 2.$$

This implies that

$$d_{C_m}(x_4) + d_{C_m}(x_2) \leqslant m - 2. \quad (2)$$

Combining (1) and (2), we have

$$d(x_2) + d(x_4) \leqslant \left|V(G-C_m)\right| + m - 2 \leqslant n - 2,$$

which contradicts the condition of Lemma 2.2.

Subcase 4.2. *$x_i x_{i+2} \notin E(G)$ for every x_i in C_m.* Observe that both x_4 and x_8 do not have any common neighbor in $G - C_m$; for otherwise, if $v \in V(G-C_m)$ is adjacent to x_4 and x_8, then we obtain a C_{m-2}: $x_4 v x_8 x_9 \ldots x_4$, which is a contradiction. Thus, we have

$$d_{G-C_m}(x_4) + d_{G-C_m}(x_8) \leqslant \left|V(G-C_m)\right|.$$

This implies that

$$\min\left\{d_{G-C_m}(x_4), d_{G-C_m}(x_8)\right\} \leqslant \left|V(G-C_m)\right|/2.$$

Without loss of generality, assume that $d_{G-C_m}(x_4) \leqslant |V(G-C_m)|/2$. Again, x_2 and x_6 do not have any common neighbor in $G - C_m$. This implies that

$$\min\left\{d_{G-C_m}(x_2), d_{G-C_m}(x_6)\right\} \leqslant \left|V(G-C_m)\right|/2.$$

Without loss of generality, assume that $d_{G-C_m}(x_2) \leqslant |V(G-C_m)|/2$. Hence we have

$$d_{G-C_m}(x_2) + d_{G-C_m}(x_4) \leqslant \left|V(G-C_m)\right|/2 + \left|V(G-C_m)\right|/2 = \left|V(G-C_m)\right|. \quad (3)$$

Thus if $x_r \in \{x_1, x_2, \ldots, x_m\} \setminus \{x_1, x_3\}$ is adjacent to x_2, then x_{r+2} is not adjacent to x_4 (for otherwise, we obtain a C_{m-2}: $x_2 x_r x_{r-1} \ldots x_4 x_{r+2} x_{r+3} \ldots x_2$, a contradiction). Observe that x_2, x_4, x_5, x_{m-1} are not adjacent to x_2, and x_4, x_6, x_7, x_1 are not adjacent to x_4, respectively. Hence we have

$$d_{C_m}(x_4) \leqslant m - \left|N_{C_m}(x_2) \setminus \{x_1, x_3\}\right| - \left|\{x_4, x_6, x_7, x_1\}\right| \leqslant m - d_{C_m}(x_2) - 2,$$

which implies that

$$d_{C_m}(x_4) + d_{C_m}(x_2) \leqslant m - 2. \quad (4)$$

Combining (3) and (4), we have $d(x_2) + d(x_4) \leqslant |V(G-C_m)| + m - 2 \leqslant n - 2$, which contradicts the condition of Lemma 2.2.

This completes the proof of Lemma 2.2. □

In order to prove the below results, we need the following Theorem 2.3 that was proved by Rao Li [3] and Shengjia Li et al. [4].

Theorem 2.3. *Let G be a 2-connected graph of order $n \geqslant 6$ such that $d(x) + d(y) \geqslant n - 1$ for each pair of nonadjacent vertices x, y in G with $d(x, y) = 2$, then G is Hamiltonian or $G \in \mathcal{C}_n \cup \mathcal{D}_n$.*

Lemma 2.4. *Let G be a 2-connected Hamiltonian graph of order $n \geqslant 6$ such that $d(x) + d(y) \geqslant n - 1$ for each pair x, y of nonadjacent vertices in G with $d(x, y) = 2$. If there is not C_{n-1} in G, then*

$$G \in \{K_{n/2,n/2}, K_{n/2,n/2} - e\},$$

where then n is an even integer.

Proof. Assume, to the contrary, that $G \notin \{K_{n/2,n/2}, K_{n/2,n/2} - e\}$. Then we have the following claims.

Claim 1. *$x_i x_{i+3} \in E(G)$ or $x_{i-1} x_{i+2} \in E(G)$ for every vertex x_i in C_n.*

Proof. Since G contains no C_{n-1}, if $x_h \in C_n$ is adjacent to x_{i+2}, then x_{h-1} is not adjacent to x_i. Namely, no vertex in $N^-_{C_n}(x_{i+2})$ is adjacent x_i. Assume that Claim 1 is not true. Then $x_i x_{i+3}, x_{i-1} x_{i+2} \notin E(G)$. Furthermore, $x_i x_{i-2}, x_{i+2} x_{i+4} \notin E(G)$ (for otherwise, if $x_i x_{i-2} \in E(G)$, then $x_{i-2} x_i x_{i+1} x \ldots x_{i-2} = C_{n-1}$, a contradiction; while if$x_{i+2} x_{i+4} \in E(G)$, then $x_{i+2} x_{i+4} x_{i+5} \ldots x_{i+2} = C_{n-1}$, a contradiction). It then can be verified that

$$\left|N_{C_n}(x_i)\right| \leqslant \left|V(G)\right| - \left|N^-_{C_n}(x_{i+2})\right| - \left|\{x_{i+3}, x_{i-2}\}\right|.$$

This implies that $d(x_i) + d(x_{i+2}) \leqslant n - 2$, which contradicts the condition of Lemma 2.4. Therefore, Claim 1 is true. □

Claim 2. *$x_i x_j \notin E(G)$ or $x_i x_{j+1} \notin E(G)$ for every pair x_i, x_j of vertices in C_n.*

Proof. If there exists x_j in C_n such that $x_i x_j, x_i x_{j+1} \in E(G)$. By Claim 1 we have $x_{i-1} x_{i+2} \in E(G)$ or $x_{i-2} x_{i+1} \in E(G)$. This implies that $x_{i-1} x_{i+2} x_{i+3} \ldots x_j x_i x_{j+1} x_{j+2} \ldots x_{i-1} = C_{n-1}$ or $x_{i-2} x_{i+1} x_{i+2} \ldots x_j x_i x_{j+1} x_{j+2} \ldots x_{i-2} = C_{n-1}$, respectively, which is a contradiction. Therefore, Claim 2 is true. □

Claim 3. *$\{x_1, x_3, x_5, \ldots, x_{2m-1}, \ldots\}$ and $\{x_2, x_4, x_6, \ldots, x_{2m}, \ldots\}$ are independent sets.*

Proof. By Claim 2, we know that $d(x_i) \leqslant n/2$ $(i = 1, 2, \ldots)$ (for otherwise, if $d(x_i) > n/2$, then there must exist $x_j, x_{j+1} \in V(C_n)$ that are adjacent to x_i, which contradicts Claim 2). Since there is no C_{n-1}, it follows that $x_{i-1} x_{i+1} \notin E(G)$, and so we have $d(x_{i-1}) + d(x_{i+1}) \geqslant n - 1$ $(i = 1, 2, \ldots)$. This implies that $(n-1)/2 \leqslant d(x_i) \leqslant n/2$ $(i = 1, 2, \ldots)$.

Then for every x_i $(i = 1, 2, \ldots)$, since there is no C_{n-1} in G, by Claim 2, there do not exist x_j, x_{j+1} in C_n such that $x_i x_j, x_i x_{j+1} \in E(G)$. Then obviously we have.

Claim 3.1. *If there exist x_h, x_{h+1} that are not adjacent to x_i, then x_i will be adjacent to every vertex of $\{\ldots, x_{h-2m+1}, \ldots, x_{h-3}, x_{h-1}, x_{h+2}, x_{h+4}, \ldots, x_{h+2m}, \ldots\}$.*

Claim 3.2. *If there do not exist x_h, x_{h+1} that are not adjacent to x_i, then when i is even, x_i will be adjacent to every vertex of $\{x_1, x_3, x_5, \ldots, x_{2m-1}, \ldots\}$; while when i is odd, x_i will be adjacent to every vertex of $\{x_2, x_4, x_6, \ldots, x_{2m}, \ldots\}$. We consider two cases.*

Case 1. Claim 3.1 does not hold. Then Claim 3 holds.

Case 2. Claim 3.1 holds. There are two subcases.

Subcase 2.1. Claim 3.2 holds. Then clearly there exist x_j, x_{j+1} such that one vertex satisfying Claim 3.1 and the other vertex satisfying Claim 3.2. This implies that x_j, x_{j+1} will be adjacent to a common vertex of C_n. It then follows by Claim 2 that we have a C_{n-1}, a contradiction.

Subcase 2.2. Claim 3.2 does not hold. In this case, by Claim 2, if x_h and x_{h+1} are not adjacent to x_i; then both x_{h+2} and x_{h+1} or both x_h and x_{h-1} are not adjacent to x_{i+1}.

In this case, by Claim 1 we have $x_{i-1}x_{i+2} \in E(G)$ or $x_i x_{i+3} \in E(G)$. Then we can obtain a C_{n-1}, a contradiction. For example, suppose that $x_{i-1}x_{i+2} \in E(G)$. Since both x_h and x_{h+1} are not adjacent to x_i, and both x_{h+2} and x_{h+1} or both x_h and x_{h-1} are not adjacent to x_{i+1}, without loss of generality, assume that both x_{h+2} and x_{h+1} are not adjacent to x_{i+1}. Then we obtain

$$C_{n-1} = x_h x_{h-1} \ldots x_{i+2} x_{i-1} x_{i-2} \ldots x_{h+2} x_i x_{i+1} x_{h+3} x_h.$$

By Case 1 and Claim 3.2, we have that $\{x_1, x_3, x_5, \ldots, x_{2m-1}, \ldots\}$ and $\{x_2, x_4, x_6, \ldots, x_{2m}, \ldots\}$ are independent sets. Therefore, Claim 3 is true. □

Since $(n-1)/2 \leqslant d(x_i) \leqslant n/2$ for $i = 1, 2, \ldots$, it follows that $G \in \{K_{n/2,n/2}, K_{n/2,n/2} - e\}$, where then n is an even integer. This completes the proof of Lemma 2.4. □

Combining Lemma 2.2, Theorem 2.3 and Lemma 2.4, we have the following main result on pancyclic graphs, which is, in fact, Theorem 1.4.

Corollary 2.5. *If G is a 2-connected graph of order $n \geqslant 6$ such that $d(x) + d(y) \geqslant n - 1$ for each pair x, y of nonadjacent vertices in G with $d(x, y) = 2$, then either G is pancyclic or*

$$G \in \mathcal{C}_n \cup \mathcal{D}_n \cup \{K_{n/2,n/2}, K_{n/2,n/2} - e\colon n \text{ is even}\}.$$

Note. Since the proof of Theorem 2.3 in [2] and [3] is very complexity. Their proofs all use the Benhocine–Wojda' Theorem in 1987, and the proof of Benhocine–Wojda' Theorem use two lemmas and Genghua Fan's Theorem in 1984, among them the proof of Lemma 2 is 4 pages. Thus, we now give a simple proof for above Theorem 2.3.

A simple proof of Theorem 2.3. Assume that G is the graph neither Hamiltonian nor $G \in \mathcal{C}_n \cup \mathcal{D}_n$ satisfying the condition of Theorem 2.3. Then let C_m be a longest cycle of G, and we have the following:

Claim 1. *If some vertex u of $G - C_m$ is adjacent to some vertex x_i in C_m, then u will be adjacent to every vertex of $G - C_m - u$.*

Otherwise, if there exists $v \in V(G - C_m - u)$ that is not adjacent to u, then v will be adjacent to x_{i+1} (otherwise, if $vx_{i+1} \notin V(G)$, since u and x_{i+1} do not have any common neighbor vertex in $G - C_m$, and clearly $d(u) = |N_{G-C_m}(u)| + |N_{C_m}(u)|$ and $d(x_{i+1}) \leqslant |N_{G-C_m}(x_{i+1})| + (|V(C_m)| - |N^+_{C_m}(u)|)$, this implies $d(u) + d(x_{i+1}) \leqslant (|N_{G-C_m}(u)| + |N_{G-C_m}(x_{i+1})|) + |N_{C_m}(u)| + (|V(C_m)| - |N^+_{C_m}(u)|) \leqslant |V(G - C_m - u - v)| + |N_{C_m}(u)| + |V(C_m)| - |N^+_{C_m}(u)| = n - 2$, a contradiction). Similarly, we have that v will be adjacent to $x_{i+3}, x_{i+5}, \ldots$, and u must be adjacent to $x_{i+2}, x_{i+4}, \ldots$, this implies $G \in \mathcal{C}_n \cup \mathcal{D}_n$, a contradiction.

Claim 2. *Let $G - C_m = H$, then $|V(H)| = 1$.*

Otherwise, if $|V(H)| > 1$, namely, $m \leqslant n - 2$. Since G is 2-connected, then for any distinct $x_{i+1}, x_{j+1} \in N^+_{C_m}(H)$ we have $d(x_{i+1}, x_{j+1}) = 2$ (otherwise, if there exist $x_{i+1}, x_{j+1} \in N^+_{C_m}(H)$ then $d(x_{i+1}, x_{j+1}) \geqslant 3$. Let $u \in V(H)$ and $x_{i+1} \in N^+_{C_m}(u)$, since both u and x_{i+1} does not have any common neighbor vertex in $G - C_m$, and clearly $x_j \notin N^+_{C_m}(u)$ and $x_j x_{i+1} \notin E(G)$, so by a similar the proof as Claim 1 we have $d(u) + d(x_{i+1}) \leqslant |V(G - C_m - u)| + |N_{C_m}(u)| + |V(C_m)| - |N^+_{C_m}(u)| - |\{x_j\}| = n - 2$, a contradiction).

Then let $x_{i+1}, x_{j+1} \in N^+_{C_m}(H)$, and $\{x_{i+1}, x_{i+2}, \ldots, x_j\} = C^+$ and $\{x_{j+1}, x_{j+2}, \ldots, x_i\} = C^-$, clearly none of $N^+_{C^+}(x_{j+1})$ are adjacent to x_{i+1}, and none of $N^-_{C^-}(x_{j+1})$ are adjacent to x_{i+1}. Thus, $d(x_{i+1}) + d(x_{j+1}) \leqslant m - (N^+_{C^+}(x_{j+1}) + N^-_{C^-}(x_{j+1}) - |\{x_{j+1}\}|) - |\{x_{i+1}\}| + |N_{C_m}(x_{j+1})| - |V(H)| \leqslant n - 2$, a contradiction.

Claim 3. $d(u) = (n-1)/2$.

Otherwise, (1) if $d(u) > (n-1)/2$. Together with Claim 2, we have that both x_{h+1}, x_{h+2} in C_{n-1} are adjacent to u, then we get a Hamiltonian cycle, a contradiction. (2) If $d(u) < (n-1)/2$. Since $d(x_{i+1}) + d(u) \geqslant n - 1$ and $d(u) + d(x_{j+1}) \geqslant n - 1$, this implies $d(x_{i+1}) > (n-1)/2$ and $d(x_{j+1}) > (n-1)/2$, so we have $d(x_{i+1}) + d(x_{j+1}) > n - 1$. Since C_m is a longest cycle, by a similar arguments as Claim 2, we have $d(x_{i+1}) + d(x_{j+1}) \leqslant n - 1$, a contradiction.

By $d(u) = (n-1)/2$, so $N^+_{C_m}(u) \cup \{u\} = \{x_{i+1}, x_{i+3}, \ldots, x_{i+2r-1}, \ldots, x_{i-1}, u\}$ is independent vertex set. Clearly for any x_{i+2r-1} $(r = 1, 2, \ldots)$, $x_i x_{i+1} x_{i+2} \ldots x_{i+2r-2} u x_{i+2r} \ldots x_i$ is also a C_{n-1}, so we can apply a similar arguments as above and get that $|N(x_{i+2r-1})| = |V(G) \setminus (N^+_{C_m}(u) \cup \{u\})| = (n-1)/2$ $(r = 1, 2, \ldots)$, this implies $G \in \mathcal{C}_n \cup \mathcal{D}_n$, a contradiction. This complete the proof of Theorem 2.3. □

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